MAT 296 — FALL 2006 — EXAM III REVIEW

Here is a section-by-section review of what we've covered since the second exam. It's an awful lot of material, so I've tried to be more chatty than is usual in a review sheet.

11.1. Sequences. A sequence of numbers $\{a_n\}$ *converges to* L if we can make a_n as close to L as we want by taking n big enough. Notation:

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty.$$

If the sequence a_n is given by some nice function $f(x)$, so $f(n) = a_n$ for every integer n, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

(Note that not every sequence is so nice: $a_n = (-1)^n$ is not of this form, since the "function" $f(x) = (-1)^x$ doesn't make sense for most non-integer values of x. Another example is the non-function $f(x) = x!$.)

This fact lets us compute limits of sequences like we do limits of functions. For example, to compute the limit of a sequence given by a *rational function* $a_n = \frac{p(n)}{q(n)}$, where the top and bottom are both polynomials in n, divide every term by the highest power of n that appears, killing off the terms of lower degree. (Note: on exam problems, writing these steps out is a very good idea.)

It also means we can apply L'Hôpital's Rule: if a sequence $\{a_n\}$ consists of fractions $f(n)/g(n)$, so both numerator and denominator are given by functions, and both $f(n)$ and $g(n)$ approach ∞ as $n \to \infty$, then $\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} f'(n)/g'(n)$.

Theorem (Squeeze Theorem for Sequences)**.** If $a_n \le b_n \le c_n$ for all n (or all n past some point), and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then b_n converges as well, and $\lim_{n\to\infty} b_n = L$.

Test (Absolute Value of a Sequence)**.** If $|a_n| \to 0$, then $a_n \to 0$ too. Note that this is only true for zero! One way to think about this is that $|a_n|$ is the *size* of a_n, so if the size of the a_n's goes to zero, then the a_n's go to zero.

Critical Example (Powers of r)**.** Fix some number r, and consider the sequence of its powers: $\{r, r^2, r^3, r^4, \ldots\} = \{r^n\}_{n=1}^{\infty}$. This sequence converges to 0 for $|r| < 1$, to 1 for $r = 1$ (duh), and diverges for all other values of r.

Theorem (Monotone Sequence Theorem)**.** If a sequence $\{a_n\}$ is either

(a) *bounded above* (so $a_n \le M$ for all n) and *increasing*, or

(b) *bounded below* (so $a_n \ge M$ for all n) and *decreasing*, then

the sequence converges.

Note that convergence is something that happens "at infinity", so changing the first term, or starting with the 34th, or adding a bunch of 17's at the beginning, makes no difference to the convergence of the sequence.

11.2. Series. An infinite sum of things $a_1 + a_2 + a_3 + \cdots$ makes no sense, usually. The way we make sense of it is to talk about the *sequence of partial sums*

$$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad \ldots \qquad s_k = a_1 + a_2 + \cdots + a_k, \qquad \ldots$$

We say the *series converges* if this sequence converges, and we write

$$\sum a_n = \lim_{k\to\infty} s_k\,.$$

Critical Example (The Geometric Series)**.** The series

$$\sum_{n=0}^{\infty} r^n \quad = \quad 1 + r + r^2 + r^3 + \cdots$$

converges for $|r| < 1$, in which case it's equal to $\dfrac{1}{1-r}$. It diverges for $|r| \ge 1$.

Critical Example (The Harmonic Series)**.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad = \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

NEVER EVER CONVERGES.

Test (Terms have to go to 0)**.** If $\lim a_n$ does not exist, or $\lim a_n \ne 0$, then the series $\sum a_n$ has no chance of converging.

11.3. The Integral Test. Now we begin a short sequence of Tests to decide whether series with all positive terms converge or not.

Test (Integral Test)**.** Let $\sum a_n$ be a series with *positive* terms, and suppose we have a continuous, positive, decreasing function $f(x)$ so that $f(n) = a_n$ for all n. Then

$$\sum_{n=0}^{\infty} a_n \text{ converges} \qquad \Longleftrightarrow \qquad \int_1^{\infty} f(x)\,dx \text{ converges.}$$

There's a picture that goes with this test (we drew it several times in class); it's probably more important to remember the picture than the proof. We don't use the Integral Test very often, but it is extremely important, if only because we used it to establish a

Critical Example (p-series)**.** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (Note the case $p = 1$ is the Harmonic Series.)

11.4. **The Comparison Tests.** Both the tests in this section, like the Integral Test, apply only to series with positive terms.

Test (Comparison Test)**.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and that $a_n \leq b_n$ for all n. Then

$$\sum b_n \text{ converges} \quad \Longrightarrow \quad \sum a_n \text{ converges,} \qquad \text{and} \qquad \sum a_n \text{ diverges} \quad \Longrightarrow \quad \sum b_n \text{ diverges}$$

Note this is just like the Squeeze Theorem for Sequences (which in turn is just like the Squeeze Theorem for Integrals). The next one, on the other hand, is new.

Test (Limit Comparison Test)**.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and set $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. If L exists and $L > 0$, then the series either both converge or both diverge. If $L = 0$, we get no information.

There are many situations where either CT or LCT will work, but LCT is a much more powerful tool. Know it, love it, live it.

11.5. **Alternating Series.** After a couple of sections about series with positive terms, we turn to *alternating* series, which feature a factor of the form $(-1)^n$ or $(-1)^{n-1}$. Oddly enough, it's much easier for an alternating series to converge.

Test (Alternating Series Test)**.** Suppose $\sum(-1)^n b_n$ is an alternating series, with the b_n all positive. If the b_n's are *decreasing* and *approach zero*, then the series converges.

So for alternating series, the Test in Section 11.2 (terms have to go to zero) is essentially enough to ensure convergence.

Critical Example (Alternating Harmonic Series)**.** In contrast to its non-alternating cousin, the Alternating Harmonic Series $\sum(-1)^{n-1}/n$ converges. In fact, as we'll see in a later section, it converges to $\ln 2$.

At this point, we had our first encounter with *estimation* of a series, and the *error* involved. It's a little deceptive, since this has nothing particularly to do with alternating series; still, for alternating series, it's especially easy. Let's back up a little and get a running start at it.

Let $\sum a_n$ be a series, and suppose that it converges: $\sum a_n = S$. This means that the partial sums $s_k = a_1 + \cdots + a_k$ converge to S. So if we wanted to get a guess at what S was, we could just take, say, the seventh partial sum! It might not be a very good guess, but it's the best we can do at the moment. Actually, the odds are very good that the eighth partial sum will be a *better* guess (in the sense that it's closer to S), the ninth will be better still, and so on, since we know $s_k \to S$ as $k \to \infty$.

So for any k, we can think of s_k as an *estimate* or *approximation* of the sum S. What's the error? Well, it's how far off we are: the distance between the sum S and our approximation s_k

$$R_k = |S - s_k|.$$

(The R is for 'remainder', which is another word for error.) Since we know $s_k \to S$, we know that $R_k \to 0$ as $k \to \infty$; the error gets smaller the more terms we add up. So by taking enough terms, we can make the error as small as we like. How many terms? Good question.

Now suppose $a_n = (-1)^n b_n$, so the series $\sum a_n$ is alternating, and assume that the b_n's decrease toward zero, so the series converges. Then we get the sum $S = \sum_{n=0}^{\infty} (-1)^n b_n$ by adding a number, subtracting a smaller one, adding a still smaller one, subtracting an even smaller one, and so on. It turns out (we drew some pictures of this) that every time we *add* a b_n, we end up above S, and every time we *subtract* one, we end up below S. In other words, the distance between any partial sum s_k and S is less than the next term, b_{k+1}:

$$R_k \le b_{k+1}.$$

This means that to estimate the sum S to within, say, 0.001, all we have to do is figure out which is the first b_n less than 0.001. Easy!

11.6. **Absolute Convergence.** Faced with the fact that the Alternating Harmonic Series converges, while the plain old vanilla Harmonic Series diverges, we think that for some series, convergence is *fragile*; it can be lost if you get rid of all the minus signs. Some series are *robust*, in that they would converge with or without their minuses (like, for example, $\sum \frac{(-1)^n}{n^2}$). The usual terms for this are *conditional* (the fragile kind) and *absolute* convergence.

Technically speaking, we say that $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. Of course, we expect absolutely convergent things to converge! So we had to prove that. Then we had two tests for absolute convergence, each of which will be extremely useful for our later work on power series.

Test (Ratio Test)**.** Suppose $\sum a_n$ is a sequence, and set

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

If L exists and is less than 1, then $\sum a_n$ converges absolutely; if L fails to exist or is bigger than 1, the series diverges; and finally, if $L = 1$ the test is inconclusive.

Test (Root Test)**.** Suppose $\sum a_n$ is a sequence, and set

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If L exists and is less than 1, then $\sum a_n$ converges absolutely; if L fails to exist or is bigger than 1, the series diverges; and finally, if $L = 1$ the test is inconclusive.

11.8. Power Series. The main motivation for our study of sequences and series has been to work our way up to *power series*, which are functions

$$f(x) = \sum c_n (x-a)^n.$$

Power series are jazzed-up polynomials, that may have terms of arbitrarily high degree. In this section and the next two, we ask two basic questions, inspired by the similarity with polynomials:

(a) What is the domain of a power series? (Remember that the domain of a polynomial is always all real numbers.)
(b) What sorts of functions can we get as power series? (That is, do we get more interesting functions than just polynomials?)

Our basic strategy for answering Question (a) was to use the Ratio Test. We want to find the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(usually as a function of x), and decide for which x-values it's less than 1. We always get one of three outcomes for a power series $f(x) = \sum c_n (x-a)^n$:

(a) $f(x)$ converges *only* for $x = a$ (this usually corresponds to getting $L = \infty$ in the Ratio Test);

(b) $f(x)$ converges for all real numbers (this happens when $L < 1$ no matter what x is);

(c) there is some number R, called the *radius of convergence*, so that $f(x)$ converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.

In the last case, notice that we get no information about what happens when $|x-a| = R$, that is, at the endpoints of the *interval of convergence*. We know that the series converges inside $(a-R, a+R)$ and diverges outside $[a-R, a+R]$, but we always have to check $a \pm R$ separately.

Notice that a power series always converges *absolutely* in $(a-R, a+R)$, but at the endpoints it may converge absolutely, conditionally, or not at all.

11.9. Representing Functions as Power Series. At the beginning, essentially the only function we really understand as a power series is the *geometric series*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

This equation is true for all x with $|x| < 1$ (but garbage for x outside that range, since the right-hand side doesn't even converge!). There are three main techniques for getting new power series representations from this one: substituting for x, differentiating, and integrating.

Differentiation and integration of a power series may seem complicated because of all the notation. It often helps to write the series $f(x) = \sum c_n(x-a)^n$ out as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

Then, for all x inside the interval of convergence, we can differentiate and integrate term-by-term:

$$\begin{aligned} f'(x) &= 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}, \end{aligned}$$

and

$$\begin{aligned} \int f(x)\,dx &= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \frac{c_3}{4}(x-a)^4 + \cdots \\ &= C + \sum_{n=0}^{\infty} \frac{1}{n+1} c_n(x-a)^{n+1}. \end{aligned}$$

Notice three things about these formulas:

(a) It doesn't really matter whether we start the sum for $f'(x)$ at $n = 0$ or $n = 1$, since the $n = 0$ term is always zero.

(b) As always, we need to toss in a "C" in the formula for $\int f(x)\,dx$; instead of putting it at the back, where it might get confused with the other c's, we usually put it in the front.

(c) If the original series had radius of convergence R, then both the derivative and the integral have radius of convergence R as well.

11.10. **Taylor and Maclaurin Series.** The three techniques for getting new power series representations from old ones, described in the previous section, are good for getting representations for a bunch of functions that are related somehow to the Geometric Series. Sadly, they're pretty much useless for dealing with, say, e^x or $\cos x$. In this section, we *assume* that a function $f(x)$ has a representation as a power series, and ask: how can we figure out what the coefficients are?

Suppose $f(x)$ is a function we care about, and let's assume that we know (somehow) that we can write $f(x) = \sum\limits_{n=0}^{\infty} c_n(x-a)^n$, a power series centered at $x = a$. (We'll talk more in the next section about when we know this.) If we don't yet know the c_n's, then we can't use the Ratio Test or any of our sophisticated tools to get information about $f(x)$. About the only thing we know is that $x = a$ is in the interval of convergence, and $f(a) = c_0$.

Oh, and we also know (from Section 11.9) how to get a power series representation centered at a for $f'(x)$. And once we know that, we can repeat to get a representation for $f''(x)$. And so on: we can write down power series representations, centered at a, for all derivatives $f^{(n)}(x)$. Since they're all centered at a, we can plug in $x = a$! Writing this out gives

$$f^{(n)}(a) = n!c_n, \qquad \text{or} \qquad c_n = \frac{f^{(n)}(a)}{n!},$$

so our original function $f(x)$ was really

$$\begin{aligned} f(x) &= \frac{f^{(0)}(a)}{0!} + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

(For this to make sense, remember that we have $0! = 1$ and $f^{(0)} = f$.)

Now, all the above was under the assumption that we knew $f = \sum_{n=0}^{\infty} c_n(x-a)^n$. We may not always know that. But even so, there's nothing stopping us from writing down the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, which is called the *Taylor series of* $f(x)$ *at* a. If $a = 0$, it gets called the

Maclaurin series for $f(x)$. Most "nice" functions are equal to their Taylor/Maclaurin series, but not all.

Critical Example (e^x, $\ln x$, $\sin x$, and $\cos x$)**.** Here are four important Taylor Series that you'll need to know (and know how to obtain!).

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \ln x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n}(x-1)^n \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

□

How to know whether a function is equal to its Taylor series? We use the idea of "estimates and errors" (see Section 11.5 above). Let $f(x)$ be some function, and let $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ be its Taylor series at $x = a$. The k^{th} partial sum of this series

$$T_k(x) = \frac{f^{(0)}(a)}{0!} + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the k^{th} *Taylor polynomial* for $f(x)$. We think of it as an approximation not just to the Taylor series, but to $f(x)$ itself! What's the remainder? It's the function $R_k(x) = f(x) - T_k(x)$.

Theorem (Taylor's Theorem)**.** Suppose that $f(x) = T_k(x) - R_k(x)$, where $T_k(x)$ is the k^{th} Taylor polynomial. If, for every x in the interval of convergence, we can show that

$$\lim_{k \to \infty} R_k(x) = 0,$$

then $f(x)$ is equal to its Taylor series.

Theorem (Taylor's Inequality)**.** We can use Taylor polynomials to approximate values of the function $f(x)$. If we know that $|f^{(k+1)}(x)| \leq M$ for all x in some interval, then

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1}.$$

The classic application of Taylor's Inequality is to estimate values of functions like sin and cos, where we know we can always take $M = 1$.