MAT 397 — FALL 2007 — EXAM II REVIEW

Note that this is not meant to be a comprehensive review. It is intended to remind you the sorts of things we've worked on, and to give you a chance to ask questions about typical problems.

The exam will cover Chapter 14. Here are some terms you should know the **precise definitions** of: $\lim_{(x,y)\to(a,b)} f(x,y) = L$, differentiability, and the directional derivative of a function at a point in the direction of a unit vector.

Here are some practice problems.

1. Find (using the $\{\cdots\}$ notation) and sketch the domains of the functions.
(a) $f(x,y) = \ln(x^2 - y)$, (b) $g(x,y) = 3x\sqrt{y} - 1$

2. Describe the graph of each function.
(a) $f(x,y) = 1 - x - \frac{1}{2}y$, (b) $g(x,y) = \sqrt{1-x^2-y^2}$, (c) $h(x,y) = -\sqrt{x^2+y^2}$

3. Describe or graph the level curves $z = k$ for $k = -2,-1,0,1,2$:
(a) $f(x,y) = x^2 + \frac{1}{4}y^2$, (b) $g(x,y) = 3x + y$, (c) $h(x,y) = y/x$

4. Find the limit or explain clearly why it does not exist.
(a) $\lim_{(x,y)\to(1,\pi/2)} \frac{2+x}{x+\cos y}$, (b) $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$
(c) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$, (d) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

5. Find the indicated partial derivatives.
(a) z_x & z_y for $z = \frac{x+y}{x-y}$, (b) $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ for $z = \cos(x^5y^4)$, (c) g_{xyy} for $g(x,y) = x^3y^5 - 2x^2y + x$
(d) All four second partial derivatives for $f(x,y) = x + \frac{y}{x}$. Is $f_{xy} = f_{yx}$? Why or why not?

6. Use the definition of differentiability to show that $f(x,y) = 2x + y^2$ is differentiable at $(1,3,11)$.

7. Find the equation of the tangent plane, *and its normal vector,* for each of the following functions at the given point.
(a) $z = x^2y$ at $(2,1,4)$, (b) $z = \sqrt{x} + \sqrt{y}$ at $(4,9,5)$, (c) $z = xe^{-2y} + y(x^2-y^2)$ at $(1,0,1)$

8. Find an equation for the plane tangent to $z = \ln(x^2+y^2)$ at $(0,1,0)$. Use a linear approximation to estimate the value of $\ln(0.82)$ by taking $\Delta x = .1, \Delta y = -.1$.

9. Suppose that $z = \sqrt{xy+y}$, $x = \cos\theta$, and $y = \sin\theta$. Find $\frac{dz}{d\theta}$ when $\theta = \pi/2$.

10. At what rate is the area of a rectangle changing if its length is 15cm and increasing at 3cm/sec while its width is 6cm and increasing at 2cm/sec?

11. Find the gradient of the function.
(a) $f(x,y) = 4x - 8y$, (b) $g(x,y) = e^{-3y}\cos 4x$, (c) $h(x,y) = y\ln(x+y)$

12. Find the directional derivative of the function at the point in the direction of the vector.
(a) $f(x,y) = (1+xy)^{3/2}$, $(3,1)$, $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$, (b) $g(x,y) = 3x^2y$, $(1,2)$, $\langle 3,4 \rangle$

13. Find and classify the critical points.
(a) $f(x,y) = x^3y + 12x^3 - 8y$, (b) $g(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

14. Find the absolute max and min of $f(x,y) = 3xy - 6x - 3y + 7$ on the triangle with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

15. Find the maximum and minimum of f subject to the constraint $g = 0$.
(a) $f(x,y) = xy$, $g(x,y) = 4x^2 + 8y^2 - 16$, (b) $g(x,y) = x - 3y - 1$, $g(x,y) = x^2 + 3y^2 - 16$

You are responsible for **the proofs of two theorems** for this exam (and the final). Here they are. If I were you, I would know them.

Theorem (The Chain Rule)**.** *Suppose that $z = f(x,y)$ is a differentiable function of x and y, and also assume that $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is a differentiable function of t, and*

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Proof. If we change t by a small amount Δt, we get changes Δx and Δy in x and y. These in turn give us a change Δz in z. As $\Delta t \to 0$, we know that $\frac{\Delta z}{\Delta t} \to \frac{dz}{dt}$, $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$, and $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$.

We know from the definition of differentiability for 2-variable functions that

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and ϵ_2 both go to zero as Δx and Δy go to zero. Divide that equation through by Δt:

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

Now since $g(t)$ and $h(t)$ are differentiable, we have

$$\Delta x = g(t+\Delta t) - g(t) \to 0 \qquad \text{and} \qquad \Delta y = h(t+\Delta t) - h(t) \to 0$$

as $\Delta t \to 0$. This forces ϵ_1 and ϵ_2 to go to 0 as Δt goes to 0. So:

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t} \\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + 0\frac{dx}{dt} + 0\frac{dy}{dt} \\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt},\end{aligned}$$

which is what we wanted to show. □

Theorem (Local Extrema)**.** *Suppose that a function $f(x,y)$ has a local minimum or local maximum at a point (a,b). If the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exist, then they are both zero.*

Proof. Define a new one-variable function $g(x)$ by $g(x) = f(x,b)$. If $f(x,y)$ has a local minimum or local maximum at $(x,y) = (a,b)$, then $g(x)$ has a local minimum or local maximum at $x = a$. From Calc I, we know that this implies $g'(a) = 0$. But the derivative $g'(x)$ is exactly the partial derivative of f with respect to x. Therefore

$$f_x(a,b) = g'(a) = 0.$$

Symmetrically, $f_y(a,b) = 0$. This is what we wanted to show. □