# MAT 631 — PROBLEM SET 2

Hand in your solution to Problems 1 and 2 by 5 pm, Wednesday, Sept 13. The other problems may be turned in at any time.

1. Let $\varphi\colon G \longrightarrow G'$ be a group homomorphism. Prove that $\varphi(x) = \varphi(y)$ if and only if $xy^{-1} \in \ker\varphi$.
2. Let $H$ be a subgroup of a group $G$, and let $g \in G$. Define

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}.$$

Prove that $gHg^{-1}$ is a subgroup of $G$, and that $H$ is normal if and only if $gHg^{-1} = H$ for all $g \in G$. (This is sometimes used as the definition of normality.)

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3. Prove that a group in which every non-identity element has order 2 is Abelian.
4. Let $\varphi\colon G \longrightarrow G'$ be a group isomorphism. Prove that the inverse function $\varphi^{-1}\colon G' \longrightarrow G$ is also an isomorphism.
5. Let $G$ be a group, and let $\varphi\colon G \longrightarrow G$ be the map $\varphi(x) = x^{-1}$. Prove that $\varphi$ is bijective, and that it is an automorphism if and only if $G$ is Abelian.
6. Describe all homomorphisms $\varphi\colon \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the additive group of integers) and determine which are injective, which are surjective, and which are automorphisms.
7. Let $H$ be a subgroup of a group $G$. Prove that the relation defined by the rule "$a \sim b$ if and only if $b^{-1}a \in H$" is an equivalence relation on $G$.