MAT 631 — PROBLEM SET 10

This problem set is for your reference only, to help you in studying for the final exam. You do not need to hand in any problems. I am available to answer questions or give hints.

1. Let A and B be real $n \times n$ matrices. Prove that if $X^tAY = X^tBY$ for all column vectors X and Y, then $A = B$. (Compare with problem 2 from Exam 1. The shorthand way to say this is that "the form determines the matrix".)

2. Let $\langle\, , \rangle$ be a symmetric bilinear form on a vector space V over a field F. Define the *quadratic form associated to* $\langle\, , \rangle$ to be the function $q\colon V \longrightarrow F$ given by $q(v) = \langle v, v\rangle$. Prove that $q(cv) = c^2q(v)$ for all $c \in F$, $v \in V$. Show that if $0 \neq 2$ in F (equivalently, 2 is invertible), then the form $\langle\, , \rangle$ can be recovered from q.

3. Let A, A' be symmetric real matrices such that $A' = PAP^t$ for some invertible P. Must A and A' have the same rank?

4. Let F be the field with two elements, and $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Use A to define a bilinear form on F^2 as usual, and prove that this form cannot be diagonalized. Define an action of $\mathrm{GL}_2(F)$ on $F^{2\times 2}$ by $g \cdot M = gMg^t$, and determine the orbits.

5. Let $V = \mathbb{R}^n$, with the usual dot product, and W a subspace. Prove that $W = W^{\perp\perp}$.

6. Let A be a complex square matrix such that XAX^* is real for all X. Prove that A is hermitian.

7. Let $\langle\, , \rangle$ be a hermitian form on a complex vector space V.

(a) Define the nullspace of the form, and check that it is a subspace of V. Define what it means for $\langle\, , \rangle$ to be nondegenerate.

(b) Define orthogonality and the orthogonal complement $W^\perp$ of a subspace W. Prove that $W^\perp$ is a subspace.

(c) If the restriction of $\langle\, , \rangle$ to W is nondegenerate, prove that $V = W \oplus W^\perp$.

8. Prove that a real symmetric matrix is positive definite if and only if its eigenvalues are all positive.

9. Let P be a real matrix which is normal and has real eigenvalues. Prove that P is symmetric.

10. Let $T\colon V \longrightarrow V$ be a linear operator on a complex vector space V. If T is either hermitian or unitary, prove that T is normal. Assume that V carries a hermitian product and an orthonormal basis B, and prove that T is hermitian (resp., unitary, normal) if and only if the matrix representing T with respect to B is hermitian (resp., unitary, normal).