MAT 632 — HOMEWORK 1

DUE ON WEDNESDAY, 23 JANUARY

1. Describe explicitly the subring of $\mathbb{C}$ generated by $\sqrt[3]{2}$ (using $\{\cdots\}$ notation).

2. Prove that $7+\sqrt[3]{2}$ and $\sqrt{3}+i\sqrt{5}$ are algebraic.

3. Let $\mathbb{Q}[\sqrt{2},\sqrt{3}]$ denote the smallest subring of $\mathbb{C}$ containing $\mathbb{Q}$, $\sqrt{2}$, and $\sqrt{3}$. Prove that $\mathbb{Q}[\sqrt{2},\sqrt{3}]=\mathbb{Q}[\sqrt{2}+\sqrt{3}]$.

4. An element r in a ring R is *nilpotent* if $r^n=0$ for some nonnegative integer n. Prove that if r is nilpotent, then $1+r$ is a unit.

5. Prove that the units of $\mathbb{C}[x]$ are precisely the constant polynomials.

6. Let R be a ring. The set of all *formal power series* $p(t)=r_0+r_1x+r_2x^2+\cdots$, with $r_i\in R$, is denoted $R[[x]]$. Prove that $R[[x]]$ is a ring. Prove that $p(t)$ is a unit of $R[[x]]$ if and only if r_0 is a unit of R.

7. Prove that fields have no proper ideals.

8. Decribe the kernel of the following ring homomorphisms.

(a) $\varphi\colon \mathbb{R}[x,y]\longrightarrow\mathbb{R}$, $\varphi(f(x,y))=f(0,0)$

(b) $\varphi\colon \mathbb{R}[x]\longrightarrow\mathbb{C}$, $\varphi(f(x))=f(3+i)$ (Hint: $3+i$ is algebraic, as shown in class.)

(c) $\varphi\colon \mathbb{Z}[x]\longrightarrow\mathbb{R}$, $\varphi(f(x))=f(\sqrt{2})$

9. Prove that the characteristic of a field F is either zero or a prime integer.

10. Let I and J be ideals of a ring R. Show by example that $I\cup J$ need not be an ideal. Prove that the smallest ideal containing both I and J is

$$I+J:=\{a+b\mid a\in I, b\in J\}.$$