MAT 732 — PROBLEM SET 2 — MODIFIED

Hand in solutions to 2 problems by Wednesday, 22 February.

(1) Prove *Schanuel's Lemma*: If

$$0 \longrightarrow N_F \longrightarrow F \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow N_G \longrightarrow G \longrightarrow M \longrightarrow 0$$

are short exact sequences of R-modules with both F and G projective, then

$$N_F \oplus G \cong \ker(F \oplus G \longrightarrow M) \cong N_G \oplus F\,,$$

where the map $F \oplus G \longrightarrow M$ is the direct sum of the two given maps $F \longrightarrow M$ and $G \longrightarrow M$. A module N_F like this is called *a first syzygy of* M, and this exercise proves that "syzygies are unique up to projective summands." (Hint: Let K be the kernel of $F \oplus G \longrightarrow M$ and find a surjective map $K \longrightarrow F$, then show that the kernel must be isomorphic to N_G.)

(2) Let

$$F_\bullet : \qquad \cdots \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a projective resolution of an R-module M. Let d be the least integer so that image$(F_d \longrightarrow F_{d-1})$ is projective. Use Schanuel's Lemma above to prove that d is independent of the projective resolution chosen, so that $d = \mathrm{pd}_R(M)$ is well-defined.

(3) Let G be a covariant functor of R-modules. We say that G is *additive* if it preserves addition of homomorphisms, so if $f, g : M \longrightarrow N$, then $G(f+g) = G(f) + G(g)$ as maps from $G(M)$ to $G(N)$. We say that G is *left-exact* if whenever

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence,

$$0 \longrightarrow G(A) \longrightarrow G(B) \longrightarrow G(C)$$

is exact as well.

(a) Prove that an additive functor takes complexes to complexes. That is, if $A \longrightarrow B \longrightarrow C$ is a complex, then so is $G(A) \longrightarrow G(B) \longrightarrow G(C)$. (This is easier than you think – just consider what happens to the zero homomorphism.)

(b) Fix an R-module M, and prove that $\mathrm{Hom}_R(M, -)$ is additive and left-exact.

(c) State (or look up) the definition of left-exactness for a contravariant functor, and prove that $\mathrm{Hom}_R(-, M)$ is left-exact too.

(4) (NEW!) Prove the "Horseshoe Lemma": Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules, and assume that we are given projective resolutions $P'_\bullet$ and $P''_\bullet$ of M' and M'', respectively. Then there exists a projective resolution $P_\bullet$ of M so that there is a short exact sequence

$$0 \longrightarrow P'_\bullet \longrightarrow P_\bullet \longrightarrow P''_\bullet \longrightarrow 0$$

is a short exact sequence of complexes. (Hint: By induction, you need only come up with P_0, and since the short exact sequence of resolutions must split in each degree, you know what P_0 is. The Snake Lemma will be useful.)

Here is a potentially helpful diagram, which also illustrates the name.