MAT 732 — WEEK 10

This week marked the

Grand Unification Moment Of The Course.

I hope you were all suitably impressed.

The players were three:

1. the vast, arid machinery of homological algebra;
2. the arcane mysteries of dimension theory á la Algebraic Geometry; and
3. the relatively elementary calculus of sequences of ring elements.

The punchline of last week's excursion into dimension theory was that one can use certain sequences of elements, so-called *systems of parameters,* to compute the dimension of a ring. A system of parameters[1]is a sequence of elements $x_1, \dots, x_d$ such that there is some maximal ideal $\mathfrak{m}$ minimal over $(x_1, \dots, x_d)$ and d is the least integer possible with this property.

A system of parameters has the property that the only prime minimal over it has maximal height. It follows that s.o.p.'s can be built inductively as follows: Since we want the primes minimal over x_1 to have height 1, we should choose x_1 to be outside all the primes of height 0. Luckily, there are only finitely many such primes (by a homework problem), and we have the Prime Avoidance Lemma. We should then choose x_2 to be outside all the minimal primes of (x_1), so that the primes minimal over (x_1, x_2) are forced to have height 2. And so on.

Modifying this process slightly leads to the idea of *regular sequences.* Instead of avoiding the (finitely many) primes *minimal* over the ideal $(x_1, \dots, x_{k-1})$ at the k^{th} step, one might avoid the (finitely many) primes *associated to* $R/(x_1, \dots, x_{k-1})$. Since we know that the set of zerodivisors on R/I is precisely the union of the associated primes, this gives the

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[1]Confession: this sentence is a lie. The cited property of systems of parameters only holds for Noetherian *local* rings. I carefully swept this distinction under the rug.

Definition 1. A sequence $\mathbf{x} = x_1, \ldots, x_d$ of elements of a ring R is *regular on an R-module M* if $\mathbf{x}M \neq M$ and for each $i = 0, \ldots, d-1$, x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$.

After a number of examples and the portentous observation that x is M-regular if and only if $M/xM \neq 0$ and the multiplication map $x\colon M \to M$ is injective, that is,

$$0 \to M \xrightarrow{x} M \to M/xM \to 0 \qquad (1)$$

is exact, we proved that permutations of regular sequences are still regular as long as they're contained in the radical of R. Then came the big moment.

Theorem 2. *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal of R with $IM \neq M$. Then I contains an M-regular sequence of length n if, and only if, $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i = 0, \ldots, n-1$.*

It continues to amaze me that injective resolutions (which one uses to define Ext) have anything do with chains of prime ideals. Just baffling.

The proof of Theorem 2 is fairly straightforward, as these things go (though I botched it slightly). The base case needs the facts we proved about associated primes (that an ideal consisting entirely of zerodivisors on M must be contained in

$$\bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$$

and prime avoidance, while the induction steps follow from basic facts about the long exact sequences of Ext obtained from (1).