MAT 732 — WEEK 12

Capitulation. As I said in last week's recap, one of my goals for the course was to avoid tensor products if at all possible. I was quite sure it could be done, except possibly for one topic: the Koszul complex. But I knew that there are (at least) two ways to define the Koszul complex other than inductively as a tensor product of complexes, so I assumed it would all turn out ok. In particular, I thought I would probably define $K_\bullet(x_1,\dots,x_n;R)$ to be the exterior algebra on the elements $x_1,\dots,x_n$.

I forgot two things. One was minor: the easy way to define the exterior algebra is to define the tensor algebra first. Hard to do without the tensor product! This can be gotten around with some strenuous gymnastics. The other was more major: proving anything about the Koszul complex is an enormous pain without using the description as a tensor product of complexes. I decided it wasn't worth it.

So we're doing tensor products. There's no shame in this. In fact, it might have gotten you (and me!) some strange looks if you'd gone around telling people you took a course in Homological Algebra but didn't know what a tensor product was or how to pronounce the word "Tor".

So here we go. After discussing R-bilinear maps briefly, we defined the tensor product $M \otimes_R N$ to be the module satisfying a certain universal property, which might be symbolized as

$$\mathrm{Hom}_R(M \otimes_R N, L) = \mathrm{Bil}_R(M,N;L)\,,$$

where $\mathrm{Bil}_R(M,N;L)$ denotes the set of R-bilinear maps $M \times N \longrightarrow L$. Since we also have a natural identifiction

$$\mathrm{Bil}_R(M,N;L) = \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,L)$$

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coming straight from the definition of bilinearity, we have here a first version of Hom $-\otimes$ *adjointness*, one of the fundamental properties of category theory.

Exercise. Let $\varphi\colon R \longrightarrow S$ be a ring homomorphism, A an R-module, and B,C two S-modules. Then each of B and C is naturally an R-module via φ.

(a) Check that $A\otimes_R B$ and $\mathrm{Hom}_R(A,B)$ are naturally S-modules, where the S-action is induced from B both times.

(b) Define $\Phi_{ABC}\colon \mathrm{Hom}_S(A\otimes_R B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B,C))$ by

$$[\Phi_{ABC}(f)(a)](b) = f(a\otimes b)$$

for $a \in A, b \in B$, and $f : A \otimes_R B \longrightarrow C$. Prove that Φ_{ABC} is an isomorphism of S-modules, with inverse $g \mapsto \{a \otimes b \mapsto g(a)(b)\}$.

(c) (Only for those with a lot of free time or serious trust issues.) Prove that Φ_{ABC} is *natural,* meaning that homomorphisms $\alpha\colon A \longrightarrow A'$, etc., induce commutative squares.

Exercise. Let $\varphi\colon R \longrightarrow S$ be a ring homomorphism and E an injective R-module. Use the Exercise above to prove in thirty seconds or less that $\mathrm{Hom}_R(S,E)$ is an injective S-module. (Hint: You want to prove that $\mathscr{F}(-) = \mathrm{Hom}_S(-, \mathrm{Hom}_R(S,E))$ is an exact functor, so takes monomorphisms to monomorphisms. The naturality property (c) above lets you.)

We proved the uniqueness of the tensor product directly from the definition, and sketched its construction. The spirit of the construction was simply to enforce bilinearity of the map $M \times N \longrightarrow M \otimes_R N$ by imposing the fewest possible relations.

Basic properties of the tensor product followed, including three that I left up to you as exercises and three that would have simplified our lives quite a bit in previous parts of the course. (No, they weren't the same three.) We then saw that the tensor product $- \otimes_R -$ is a functor of two arguments, like $\mathrm{Hom}_R(-,-)$. In particular, fixing one argument gives a (covariant) functor from R-modules to R-modules, which we proved was right-exact.

At this point, as I mentioned in class, the machinery we built in the first few weeks of the course was poised to take over and rumble along the road to Tor. The process would have been something like this: To compute $\mathrm{Tor}_i^R(M,N)$, let $P_\bullet$ be a projective resolution of M, apply $-\otimes_R N$, and take homology. Of course, one might also want to let $Q_\bullet$ be a projective resolution of N, apply $M\otimes_R -$, and take homology; it is a theorem that these two approaches give the same answer. In particular, just as for Hom and Ext, we have $\mathrm{Tor}_0^R(M,N)\cong M\otimes_R N$. Since $M\otimes_R N\cong N\otimes_R M$, it is not hard to see that $\mathrm{Tor}_i^R(M,N)\cong \mathrm{Tor}_i^R(N,M)$ for every i.

Standard facts: $\mathrm{Tor}_i^R(M,R)=0$ for all $i\geq 1$, and a short exact sequence $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ leads via the Snake Lemma to a long exact sequence of Tor:

$$\cdots\longrightarrow \mathrm{Tor}_i^R(M,B)\longrightarrow \mathrm{Tor}_i^R(M,C)\longrightarrow \mathrm{Tor}_{i+1}^R(M,A)\longrightarrow\cdots.$$

Exercise. Let $x\in R$ be a nonzerodivisor. Show that

$$\mathrm{Tor}_1^R(R/(x),M)=\{m\in M\mid xm=0\}$$

and connect this with a lemma from last week.

Exercise. Let I,J be two ideals of R. Observe that $IJ\subseteq I\cap J$, but we can have strict containment (if $I=J$ for example). We showed in class that $R/I\otimes_R R/J\cong R/(I+J)$. Prove that

$$\mathrm{Tor}_1^R(R/I,R/J)\cong (I\cap J)/IJ\,.$$

To round out the week, we defined the tensor product of two complexes (a straight generalization of the notion just for modules). Our main example (and, as I pointed out, the justification for all this) is going to be the Koszul complex, which we'll investigate in detail next week.