# MAT 732 — WEEK 5

Functors took center stage this week. After finishing the proof of the Comparison Theorem for Projective Resolutions, I defined what it means for a functor $F$ to be additive, covariant/contranvariant, and left/right exact. Our standard example for most of the week was $\mathrm{Hom}_R(-, N)$ for some fixed $R$-module $N$, which is additive, contravariant, and left-exact. For any functor like this, we defined the *right derived functors* of $F$, $R^nF(-)$, by the following recipe: to compute $R^nF(M)$, take a projective resolution of $M$, apply $F$ to obtain a complex, and take the homology at the $n^{\text{th}}$ module[1].

We computed several examples in the case $F(-) = \mathrm{Hom}_R(-, N)$, in which case $R^nF(-)$ is denoted by $\mathrm{Ext}^n_R(-, N)$. In particular, we showed that $\mathrm{Ext}^0_R(-, N) = \mathrm{Hom}_R(-, N)$ and that $\mathrm{Ext}^n_R(R, N) = 0$, and we computed $\mathrm{Ext}^n_R(R/(x), N)$ when $x$ is a nonzerodivisor of $R$. In order to do this last one, we had to show that Hom was a *multiplicative* functor, so applying it to a homomorphism $M \xrightarrow{x} M$, multiplication by a ring element $x$, gives $F(M) \xrightarrow{x} F(M)$, multiplication again.

Finally, we defined the *Betti numbers* of a module $M$ over a local ring $(R, \mathfrak{m}, k)$ to be the ranks of the free modules in a minimal free resolution of $M$, and we showed that they are precisely the dimensions of $\mathrm{Ext}^n_R(M, k)$ as vector spaces.

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[1] There was a little ambiguity about this, since applying $F$ flips the complex around so that the differentials *raise* degree, rather than lowering it, so where exactly is the $n^{\text{th}}$ module? The technically correct way to manage this is to realize that since its differential raises degree, $F(P_\bullet)$ is now a *co*chain complex. To differentiate cochain complexes from chain complexes, we raise their indices from subscripts to superscripts. Then we're really taking the *co*homology at the $n^{\text{th}}$ module; cohomology is just homology with the indices raised. (This isn't really true – among other things, cohomology has products and homology doesn't – but it's good enough for our purposes. In any case, it's just kernels mod images.) Among other things, this explains why we write $R^nF(-)$ instead of $R_nF(-)$.

For general functors $F$ (still additive, contravariant, left-exact), we proved that $R^nF(M)$ does not depend on the choices you make in building a projective resolution of $M$. This was a straightforward application of the Comparison Theorem (and is in fact the reason we proved the CT). Other general facts: $R^nF(-)$ is again an additive contravariant functor (though no longer left-exact!), and there is a long exact sequence:

**Proposition.** *For any short exact sequence*

$$0 \to A \to B \to C \to 0\,,$$

*there is a long exact sequence*

$$\begin{aligned} 0 \to R^0F(C) \to R^0F(B) \to R^0F(A) \xrightarrow{\delta} R^1F(C) \to \cdots \\ \cdots \to R^nF(B) \to R^nF(A) \xrightarrow{\delta} R^{n+1}F(C) \to R^{n+1}F(B) \to \cdots \end{aligned}$$

*in which the unlabeled arrows are induced by the given exact sequence, and the* $\delta s$ *are called "connecting homomorphisms."*

The proof of this was an application of the Horseshoe Lemma (which is on your current problem set) and the long exact sequence of homology (see week 3).

Some indecision slowed us down here. I decided to go ahead and discuss other flavors of derived functors (first for covariant left-exact functors, and later right-exact ones of both variances). As a first step, we discussed the procedure if you're given $F$ that is additive, left-exact, and covariant: to compute $R^n(M)$, take an *injective* resolution of $M$, apply $F$ to get a complex, and take the $n^{\text{th}}$ homology[2]. As an example, I pointed out $\mathrm{Hom}_R(M, -)$ for a fixed $R$-module $M$. In this case, the right-derived functors are denoted $\mathrm{Ext}^n_R(M, -)$. Lucky for us, Ext = Ext, a fact that I asserted but which is beyond us for now. The jargon for this is that "Ext is a *balanced* functor of two arguments."

In order to really understand what this means, we need to talk about injective modules. I defined them by reversing all the arrows in the definition of projective modules, and we started to discuss the theory

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[2] actually cohomology, see footnote 1

of injective Abelian groups. Interestingly, that's where all the action is — every ring inherits its injective modules from $\mathbb{Z}$. We'll prove this next.

I'd like to take this opportunity to emphasize something that fell somewhat by the wayside in class (it's hard to keep a global vision of the course when we meet only twice a week). Recall that a functor $F$ is called *exact* if $F$ takes exact sequences to exact sequences.

**Fact 1.** *Let $R$ be a commutative ring and $M$, $N$ two $R$-modules. Then:*

1. *The (covariant) functor $\operatorname{Hom}_R(M,-)$ is left-exact, and is exact if and only if $M$ is a projective $R$-module.*
2. *The (contravariant) functor $\operatorname{Hom}_R(-,N)$ is left-exact, and is exact if and only if $N$ is injective.*

We've proven all the pieces of this, but never put them all together in the same place. To connect up with the derived functors, here's a fact that we'll prove next week. Assuming that Ext is balanced, it follows directly from stuff we've done.

**Fact 2.** *Let $R$ be a commutative ring and $M$ an $R$-module. Then*

1. *The module $M$ is injective if and only if $\operatorname{Ext}^i_R(-,M)=0$ for all $i \geq 1$, and this happens if and only if $\operatorname{Ext}^1_R(-,M)=0$.*
2. *The module $M$ is projective if and only if $\operatorname{Ext}^i_R(M,-)=0$ for all $i \geq 1$, and this happens if and only if $\operatorname{Ext}^1_R(M,-)=0$.*

Since we can only do so much in this class, I want to use this space to point out some more examples of derived functors, just so you don't think we've exhausted the subject.

(a) The torsion functors Tor, which we'll define and study next week.

(b) Local cohomology $H^n_I(M)$ of a module $M$ with support in an ideal $I$. These are the right derived functors of the "$I$-torsion" functor $\Gamma_I(-)$, defined by

$$\Gamma_I(N) = \{x \in N \mid I^t x = 0 \text{ for some } t \geq 0\}\,.$$

To compute, take an injective resolution $I^\bullet$ of $M$, apply $\Gamma_I(-)$, and take cohomology.

(c) Sheaf cohomology $H^i(X, \mathcal{F})$ for a topological space $X$ and a sheaf $\mathcal{F}$. These are the right derived functors of the "global sections" functor that associates to $\mathcal{F}$ the abelian group $\mathcal{F}(X)$, and gets computed just like local cohomology.

(d) Group cohomology $H^i(G, M)$ of a group $G$ with coefficients in a $G$-space $M$. These are the right derived functors of the "fixed-points" functor $M \mapsto M^G$.

(e) There are also apparently derived categories used in functional analysis, but I don't know anything about this. Google is your friend.