MAT 732 — WEEK 7

This week we began to apply all the abstraction of derived functors &c., which we worked so hard to achieve, to actual commutative ring theory. Our first application of our work on Ext was to prove Auslander's theorem [1]:

Theorem 1 (Auslander)**.** *The following conditions are equivalent for a ring R and an integer $n \geq 0$:*

(a) $\operatorname{pd}_R M \leq n$ *for every R-module M;*
(b) $\operatorname{pd}_R R/J \leq n$ *for every ideal J of R;*
(c) $\operatorname{id}_R N \leq n$ *for every R-module N;*
(d) $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i \geq n+1$ and every pair of R-modules M, N.*

With this theorem as transition, we began to work our way toward a careful study of finitely generated modules over Noetherian local rings (a.k.a. "the promised land"). Preparatory to this, we needed to prove some of the basic results for finite generation, Noetherianness, and locality. For example, we proved the Cayley–Hamilton theorem for arbitrary rings, and derived from it Nakayama's Lemma (NAK). Using NAK, we showed that (1) the minimal number of generators of a module, while not well-defined in general, makes sense over local rings, and (2) finitely generated projective modules over local rings are free.

References

[1] M. Auslander , *On the dimension of modules and algebras (III), Global dimension*, Nagoya Math. J. **9** (1955), 67–77. [1]

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[1]Historical note: this was Auslander's first published article, written just after he got his degree. It's odd, therefore, that it should be part 'III' in a series. In fact, there were ten papers titled *On the dimension of modules and algebras* in the Nagoya Mathematics Journal between 1955 and 1958, by various subsets of: Eilenberg, Ikeda, Nakayama, Auslander, Nagao, Jans, Rosenberg, Zelinsky, Beršteĭn, and Kaplansky. Auslander was responsible for III and VI. The book *Homological Algebra* by Cartan and Eilenberg had just appeared, and homological methods were very sexy. The next development, which we'll come to in due course, was the series of papers by Auslander and Buchsbaum.