MAT 732 — WEEK 8

Continuing our progress toward the careful study of modules over Noetherian local rings, we began this week with a whirlwind review of/introduction to localization. Since we're mainly interested in localization as a machine that produces local rings, we only covered the case of localization *at a prime ideal* $\mathfrak{p}$. In fact, what we needed was merely that *the complement* $R \setminus \mathfrak{p}$ *is multiplicatively closed*, in the sense that if $r, s \in R \setminus \mathfrak{p}$, then the product rs is as well. This follows directly from the definition of a prime ideal, but there are other examples as well:

Examples.

(1) Let $x \in R$ be any element. Then the set $\{1, x, x^2, \dots\}$ of all powers of x is multiplicatively closed. The localization R_x is obtained by inverting x (and so necessarily inverting all powers of x).

(2) The set of all nonzerodivisors in R is multiplicatively. More generally (see below), if $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ are prime ideals, then the complement of the union $S = \bigcup_{i=1}^{t} \mathfrak{p}_i$ is multiplicatively closed. The localization $S^{-1}R$ is a *semilocal* ring, meaning that it has only finitely many maximal ideals, namely the extensions $\mathfrak{p}_i S^{-1}R$.

After a few examples of localizations, we proved the "local-global principle": An element of a module is zero if and only if it maps to zero in all localizations.

The set of all primes of R is denoted $\operatorname{Spec}(R)$. Three sets of primes attached to a module M are of interest to us: the set $\operatorname{Min}_R(M)$ of prime ideals minimal over $\operatorname{ann}_R M$, the set $\operatorname{Ass}_R(M)$ of primes associated to M, and the set $\operatorname{Supp}_R(M)$ of primes $\mathfrak{p}$ such that the localization $M_\mathfrak{p}$ is nonzero. We proved that $\operatorname{Min}_R(M) \subseteq \operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$. In the special case $M = R/I$ for some ideal I, this says that the primes minimal over I are prime divisors of I, and every prime divisor of I contains I.

The standard fact that $\operatorname{nilrad}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$ led us to classify the zerodivisors of R similarly: They consist of the union of the associated primes of R. Otherwise said, the nonzerodivisors of R are the complement of the union of its associated primes.

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The last ingredient we need before cooking up a big mess of commutative algebra stew is the Hilbert Basis Theorem:

Hilbert Basis Theorem. *For a Noetherian ring R, the polynomial extension $R[x]$ is again Noetherian.*

We gave a proof of HBT using ideas from Gröbner basis theory. Hilbert's original proof was non-constructive, meaning that it did not explicitly produce a finite generating set for a chosen ideal of $R[x]$, but merely showed that one must exist. The story that goes with this is that Gordan rejected Hilbert's paper for publication, exclaiming, "das ist keine Mathematik, das ist Theologie!" ("This is not Mathematics, this is Theology!"). Gordan was the reigning king of invariant theory – that is, of all of algebra – at the time, and worked exclusively algorithmically. His methods eventually took a back seat to those of Hilbert and of Gordan's only student, Emmy Noether.

Finally, we saw that while a common-sense, follow-your-nose attempt at proving HBT doesn't quite work, it does prove that the ring of *formal power series* $R[\![x]\!]$ over R is Noetherian if R is.