MAT 732 — WEEK 9

This week and part of next are devoted to studying the first steps in dimension theory. The Krull dimension of a commutative ring is the most common measure of its size, and the one that jibes best with the connections in algebraic geometry. After defining the height of a prime ideal and, from it, the dimension of a ring, and working through a few examples, we stated but did not prove:

Theorem. *For an arbitrary Noetherian ring R,* $\dim R[x] = \dim R + 1$.

We will actually put together all the tools we need to prove this by early next week. The first steps in dimension theory are frustrating, because one has the choice of either doing a lot of theory immediately so that you can work examples out to the end, or working examples early and only being able to give bounds, rather than exact answers. We were able to work through most of the details in Nagata's example of a Noetherian ring of infinite Krull dimension.

After a few basic results on dimension, we focused on the case of dimension zero, and proved the foundational fact that a Noetherian ring is 0-dimensional if and only if it is Artinian.

In the bonus class on Friday (making up an earlier cancelled class) we discussed

The Principal Ideal Theorem of Krull. *If a prime ideal in a Noetherian ring is minimal over an n-generated ideal, then it has height at most n.*

This necessitated the introduction of *symbolic powers* of a prime ideal, which are interesting in their own rights. It also enabled us to introduce an equivalent notion of dimension: the dimension of R is the minimal number of elements in a *system of parameters* for R, that is, a sequence of elements generating an ideal over which a maximal ideal is minimal. Finally, we were able to conclude the Theorem above about the dimension of polynomial extensions.

The proof of a converse for Krull's PIT needed the Prime Avoidance Lemma, which I skipped proving in class, so here it is.

The Prime Avoidance Lemma. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of a Noetherian ring R, and I an ideal such that $I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $I \subseteq \mathfrak{p}_i$ for some i.*

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Proof. The case $n = 1$ is trivial. For $n \geq 2$, we may assume for induction that I is not contained in the union of any $n-1$ of the $\mathfrak{p}_i$. In particular, for each $j = 1, \ldots, n$ there exists $x_j \in I$ such that $x_j \notin \bigcup_{i \neq j} \mathfrak{p}_i$. Since $x_j \notin \mathfrak{p}_i$ for all $i \neq j$, we must have $x_j \in \mathfrak{p}_j$.

Assume now that $n = 2$. Then $x_1 \in \mathfrak{p}_1$ and $x_2 \notin \mathfrak{p}_1$, so the sum $x_1 + x_2 \notin \mathfrak{p}_1$. Symmetrically, $x_1 + x_2 \notin \mathfrak{p}_2$. But this contradicts $I \subseteq p_1 \cup \mathfrak{p}_2$.

For $n > 2$, the same argument with $x_1 + x_2 x_3 \cdots x_n$ gives the same contradiction. ☐

It's somewhat surprising that Prime Avoidance can be jazzed up as follows (this is the version in Eisenbud's book, Lemma 3.3):

The Jazzed-Up Prime Avoidance Lemma. *Suppose* $I_1, \ldots, I_n, J$ *are ideals of a ring* R*, and suppose that* $J \subseteq \bigcup_{j=1}^n I_j$*. If* R *contains an infinite field or if at most two of the* I_j *are not prime, then* J *is contained in one of the* I_j*.*

It's even more surprising that there are applications in this generality.

Note in particular that the Noetherian hypothesis is not needed, which answers a question Christy asked.