4

Invariant Theory

Historical jabber.

We begin with a little general invariant theory.

§1 Group actions on power series rings

We intend to investigate *invariant subrings* of elements of a power series ring $S = k[[x_1,\dots,x_n]]$ left immobile by the action of a finite group of ring automorphisms $G \subseteq \mathrm{Aut}(S)$. Denote this fixed ring by S^G. First we observe that we may assume the action of G is linear on the variables x_i. This argument goes back to Cartan [Car57].

4.1 Lemma. *Let k be a field, $S = k[[x_1,\dots,x_n]]$ a power series ring over k, and $G \subseteq \mathrm{Aut}(S)$ a finite group of ring automorphisms of S with $|G|$ invertible in k. Then there exists a finite group $G' \subseteq \mathrm{GL}(n,k)$, acting on S via linear changes of variable such that $S^{G'} \cong S^G$.*

Proof. Let $V = (x_1,\dots,x_n)/(x_1,\dots,x_n)^2$ be the vector space of linear forms of S. Then G acts on V, giving a group homomorphism $\rho\colon G \longrightarrow \mathrm{GL}(V)$. Set $G' = \rho(G)$, and extend the action of G' linearly to all of S. For $\sigma \in G$, denote by $\overline{\sigma}$ its image in G'.

Define a ring homomorphism $\theta\colon S \longrightarrow S$ by the rule

$$\theta(f) = \frac{1}{|G|}\sum_{\sigma\in G} \overline{\sigma}^{-1}\sigma(f).$$

Since θ restricts to the identity on V, it is an automorphism of S. For an invariant $f \in S^G$, $\theta(f)$ is the average of the G'-orbit of f, so is invariant under the action of G'. Restricting θ to S^G thus delivers the isomorphism $S^G \cong S^{G'}$. □

4.2 Notation. We adopt the following notation for the entire chapter. Let k be a field, and fix the power series ring $S = k[\![x_1,\dots,x_n]\!]$ of dimension n over k. Let $\mathfrak{n} = (x_1,\dots,x_n)$ be the maximal ideal of S, and $V = \mathfrak{n}/\mathfrak{n}^2$ the vector space of linear forms. Let G be a finite subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}(n,k)$, acting linearly on S, and assume that $|G|$ is nonzero in k. Set $R = S^G$, the invariant ring.

4.3 Remarks.

1. The reader with too little to do may replace power series throughout by $k\{x_1,\dots,x_n\}$, a ring of convergent power series over a valued field k. This local ring is Henselian, so by Theorem 1.4 satisfies the Krull-Remak-Schmidt property. We will stick to the complete case, deferring to Chapter 10 questions of ascent and descent.

2. When $n = 1$, R is again a regular local ring. We consider this situation dull, and rule it out from now on.

3. The assumption that $|G| \neq 0$ in k is essential for what we do below; virtually everything breaks terribly in the "modular" situation. Since we do insist upon it, we have an R-linear *Reynolds operator* $\rho\colon S \longrightarrow R$, defined by sending $f \in S$ to the average of its orbit:
$$\rho(f) = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(f).$$
This splits the natural inclusion $R \subseteq S$, thereby making R an R-direct summand of S. It follows that $IS \cap R = I$ for every ideal I of R, so

that R is Noetherian, local, and even complete, with maximal ideal $\mathfrak{m} = (x_1,\dots,x_n) \cap R$. (The Reynolds operator is not strictly necessary for the Noetherian property, but it simplifies matters significantly and will definitely be needed below.)

4. Let K be the quotient field of S and F the quotient field of R. Then G acts naturally on K, and it's easy to see that $R = S \cap F$. This implies that R is a normal domain. Now every element $f \in S$ is a root of the monic polynomial $\prod_{\sigma \in G}(X - \sigma(f))$, whose coefficients are elementary symmetric polynomials in the conjugates $\{\sigma(f)\}$. This shows that S/R is an integral extension, and in particular $\dim R = \dim S = n$. Furthermore, the field extension K/F is separable algebraic. The Primitive Element Theorem and a quick argument on degrees then show that $F = K^G$, so that in particular K/F is a finite Galois extension, and S is a finitely generated R-module.

5. The presence of the Reynolds operator and the fact that S/R is integral allows us to invoke the Hochster–Eagon theorem [HE71], [BH93, Theorem 6.4.5] to conclude that R is a Cohen–Macaulay ring. In fact this is overkill: the equality $IS \cap R = I$ for ideals I of R already implies Cohen–Macaulayness.

To understand the subring R, we move to an overring of S, and a mildly non-commutative one at that.

4.4 Definition. Let $S\#G$ denote the *skew group ring* of S and G. As an S-module, $S\#G$ is free on the elements of G; the product of two elements

$s_1\sigma_1$ and $s_2\sigma_2$ is

$$(s_1\sigma_1)(s_2\sigma_2) = s_1\sigma_1(s_2)\sigma_1\sigma_2\,.$$

(Thus moving σ_1 past s_2 "twists" the ring element. See the paragraph before 4.15 below for an explanation of this odd rule.)

4.5. Let M be an S-module. Then a left $S\#G$-module structure on M is nothing but an action of G which is compatible with that of S, in the sense that $\sigma(sm) = \sigma(s)\sigma(m)$ for all $\sigma \in G$, $s \in S$, $m \in M$. Since the action of G on S is defined on the variables and extended linearly, we have $\sigma(st) = \sigma(s)\sigma(t)$, and so S itself is a $S\#G$-module.

Similarly, an $S\#G$-linear map between left $S\#G$-modules is an S-module homomorphism $f\colon M \longrightarrow N$ respecting the action of G, so that $f(\sigma(m)) = \sigma(f(m))$. This allows us to define a left $S\#G$-module structure on $\operatorname{Hom}_S(M,N)$, when M and N are $S\#G$-modules, by $\sigma(f)(m) = \sigma(f(\sigma^{-1}(m)))$. It follows that an S-linear map $f\colon M \longrightarrow N$ between $S\#G$-modules is $S\#G$-linear if and only if it is invariant under the G-action. Indeed, if $\sigma(f) = f$ for all $\sigma \in G$, then $f(m) = \sigma(f(\sigma^{-1}(m)))$, so that $\sigma^{-1}(f(m)) = f(\sigma^{-1}(m))$ for all $\sigma \in G$. Concisely,

$$\operatorname{Hom}_{S\#G}(M,N) = \operatorname{Hom}_S(M,N)^G\,. \tag{4.5.1}$$

Since the order of G is invertible in k, taking G-invariants of an $S\#G$-modules is an exact functor[1]. In particular, $-^G$ commutes with taking co-

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[1] To see this, first note that $-^G$ is clearly left-exact. Then for an $S\#G$-linear surjection $f\colon M \longrightarrow N$, and $n \in N^G$, note that $f(\sigma(m)) = \sigma(f(m) = \sigma(n) = n$ for every preimage $m \in M$ of n. Taking the average of the orbit of such preimages, then, gives an element of M^G mapping to n.

homology, so (4.5.1) extends to higher Exts:

$$\operatorname{Ext}^i_{S\#G}(M,N) = \operatorname{Ext}^i_S(M,N)^G \tag{4.5.2}$$

for all $S\#G$-modules M and N and all $i \geqslant 0$. This has the following wonderful consequence, the easy proof of which we leave as an exercise.

4.6 Proposition. *An $S\#G$-module M is projective if and only if it is projective (that is, free) as an S-module. In particular, the twisted group ring $S\#G$ has finite global dimension, equal to n.*

The next example may come in handy when proving the last assertion of the Proposition.

4.7 Example. The Koszul complex $K_\bullet$ on the sequence of variables $\mathbf{x} = x_1,\dots,x_n$ is a minimal $S\#G$-linear resolution of the residue field k of S (with trivial action of G). In detail, let $V = \mathfrak{n}/\mathfrak{n}^2$ again be the k-vector space with basis $x_1,\dots,x_n$, and

$$K_p = K_p(\mathbf{x},S) = S \otimes_k \bigwedge^p V$$

for $p \geqslant 0$. The differential $\partial_p : K_p \longrightarrow K_{p-1}$ is given by

$$\partial_p(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} x_{i_j}(x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_p}),$$

where $\{x_{i_1} \wedge \cdots \wedge x_{i_p}\}$, $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$, are the natural basis vectors for $\bigwedge^p V$. Since the x_i form an S-regular sequence, K_p is acyclic, minimally resolving k.

The exterior powers $\bigwedge^p V$ carry a natural action of G, by $\sigma(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sigma(x_{i_1}) \wedge \cdots \sigma(\wedge x_{i_p})$, and it's easy to see that the differentials ∂_p are $S\#G$-linear for this action. Since the modules appearing in $K_\bullet$ are free

S-modules, they are projective over $S\#G$, so we see that $K_\bullet$ resolves the trivial module k over $S\#G$. Since every projective over $S\#G$ is free over S, the Depth Lemma then shows that $\operatorname{pd}_{S\#G} k$ cannot be any smaller than n.

The module theory of the skew group ring faithfully reflects the representation theory of G, in a precise sense.

4.8 Definition. Let M be an $S\#G$-module and W a k-representation of G, that is, a module over the group algebra kG. Define an $S\#G$-module structure on $M \otimes_k W$ by the diagonal action

$$s\sigma(m \otimes w) = s\sigma(m) \otimes \sigma(w).$$

Define a functor $\mathscr{F}$ from the category of finite-dimensional k-representations W of G to that of finitely generated $S\#G$-modules by

$$\mathscr{F}(W) = S \otimes_k W$$

and similarly for homomorphisms. For any W, $\mathscr{F}(W)$ is obviously a free S-module and thus a projective $S\#G$-module.

In the opposite direction, let P be a finitely generated projective $S\#G$-module. Then $P/\mathfrak{n}P$ is a finite-dimensional k-vector space with an action of G, that is, a k-representation of G. Define a functor $\mathscr{G}$ from projective $S\#G$-modules to k-representations of G by

$$\mathscr{G}(P) = P/\mathfrak{n}P$$

and correspondingly on homomorphisms.

4.9 Proposition. *The functors $\mathscr{F}$ and $\mathscr{G}$ form an adjoint pair, that is,*

$$\operatorname{Hom}_{kG}(\mathscr{G}(P), W) = \operatorname{Hom}_{S\#G}(P, \mathscr{F}(W)),$$

and are inverses of each other on objects. Concretely, for a projective $S\#G$-module P and a k-representation W of G, we have

$$S \otimes_k P/\mathfrak{n}P \cong P$$

and

$$(S \otimes_k W)/\mathfrak{n}(S \otimes_k W) \cong W\,.$$

In particular, there is a one-one correspondence between the isomorphism classes of indecomposable projective $S\#G$-modules and the irreducible k-representations of G.

Proof. It is clear that $\mathscr{G}(\mathscr{F}(W)) \cong W$, since

$$(S \otimes_k W)/\mathfrak{n}(S \otimes_k W) \cong S/\mathfrak{n} \otimes_k W \cong W\,.$$

To show that the other composition is also the identity, let P be a projective $S\#G$-module. Then $\mathscr{F}(\mathscr{G}(P)) = S \otimes_k P/\mathfrak{n}P$ is a projective $S\#G$-module, with a natural projection onto $P/\mathfrak{n}P$. Of course, the original projective P also maps onto $P/\mathfrak{n}P$. This latter is in fact a projective cover of $P/\mathfrak{n}P$ (since idempotents in kG lift to $S\#G$ via the retraction $kG \longrightarrow S\#G \longrightarrow kG$). There is thus a lifting $S \otimes_k P/\mathfrak{n}P \longrightarrow P$, which is surjective modulo $\mathfrak{n}P$. Nakayama's Lemma then implies that the lifting is surjective, so split, as P is projective. Comparing ranks over S, we must have $S \otimes_k P/\mathfrak{n}P \cong P$. $\square$

4.10 Corollary. *Let* $V_0, \ldots, V_d$ *be a complete set of non-isomorphic simple* kG*-modules. Then*

$$S \otimes_k V_0, \ldots, S \otimes_k V_d$$

is a complete set of non-isomorphic indecomposable finitely generated projective $S\#G$*-modules. Furthermore, the category of finitely generated projective* $S\#G$*-modules satisfies the Krull–Remak–Schmidt property, i.e. each finitely generated projective* P *is isomorphic to a unique direct sum* $\bigoplus_{i=0}^{d}(S \otimes_k V_i)^{n_i}$.

§2 The endomorphism algebra

We maintain the notation of 4.2. The results of the previous section establish a one-one correspondence between the isomorphism classes of finitely generated indecomposable projective modules over the skew group ring $S\#G$ and the irreducible k-representations of G. Here we extend this correspondence to the finitely generated indecomposable $\operatorname{End}_R(S)$-modules (by the simplest means possible, namely, proving that $\operatorname{End}_R(S) \cong S\#G$), and further to the indecomposable direct summands of S as an R-module. Each of these summands is of course a MCM R-module. In general, we do not get all the indecomposable MCM R-modules in this way, but see the next chapter for the case of dimension two.

Let us say a word about the correspondence between $S\#G$-modules and direct summands of S. Consider the S-module homomorphism $\rho\colon S \longrightarrow S\#G$ defined by

$$\rho(s) = \sum_{\sigma\in G} \sigma(s)\, \sigma\,.$$

It's easy to see that ρ is in fact an injective ring homomorphism, with image equal to the fixed points $(S\#G)^G$. In particular, we have

$$(S\#G)^G = \rho(S) \cong S\,.$$

Extending this to direct summand of direct summands of S gives a functor

$$\mathscr{H}\colon \operatorname{add}_{S\#G}(S\#G) \longrightarrow \operatorname{add}_R(S)\,.$$

Of course, $\operatorname{add}_{S\#G}(S\#G)$ is exactly the category of finitely generated projective $S\#G$-modules.

We will show that under an additional assumption on G, the functor $\mathscr{H}$ is an equivalence of categories. See Theorem 4.17. We turn now to this additional assumption.

4.11 Definition. An element $\sigma \in \mathrm{GL}(V)$ of finite order is called a *pseudo-reflection* provided $\sigma - \mathrm{id}$ has rank 1. A pseudo-reflection σ is a *reflection* if it has order 2. We say a subgroup $G \subseteq \mathrm{GL}(V)$ is *small* if it contains no pseudo-reflections but the identity.

If a non-identity pseudo-reflection σ is diagonalizable, then σ is similar to a diagonal matrix with diagonal entries $1,\dots,1,\lambda$ with $\lambda \neq 1$ a root of unity. Geometrically, σ is a pseudo-reflection if it has finite order and its fixed set $V^\sigma = \{x \in V \mid \sigma(x) = x\}$ has codimension one.

The importance of pseudo-reflections to our problem begins with the foundational theorem of Shephard–Todd [BH93, 6.4.12].

4.12 Theorem (Shephard–Todd, Chevalley, Serre)**.** *Let k be a field and $S = k[[x_1,\dots,x_n]]$ a power series ring over k. Let $G \subseteq \mathrm{GL}(n,k)$ be a finite* *group generated by pseudo-reflections, acting linearly on S. Then S^G is a complete regular local ring, namely, a power series ring $k[[z_1,\dots,z_n]]$ in algebraically independent elements $z_1,\dots,z_n$.*

In fact, by a Theorem of Prill, we may always assume that G is small. See Appendix B for this, which will we will not use in this chapter.

It is clear that $\mathrm{SL}(n,k)$ contains no non-trivial pseudo-reflections, so that every finite subgroup of $\mathrm{SL}(n,k)$ is small. Moreover, we have the following result of Watanabe, the proof of which would take us too far afield.

4.13 Proposition (Watanabe)**.** *Assume $G \subseteq \mathrm{GL}(n,k)$ is small. Then $R = S^G$ is Gorenstein if and only if $G \subseteq \mathrm{SL}(n,k)$.*

See the next chapter for our best application of this result.

To connect the smallness of the group G with properties of the twisted group ring $S\#G$, we must pause for a moment's contemplation of ramification theory. We banish all the details to Appendix B.

Let A be a Noetherian ring and B a commutative, finitely generated A-algebra. A prime ideal $\mathfrak{P} \in \operatorname{Spec} B$ is *unramified* (over A, or over $\mathfrak{p} = \mathfrak{P} \cap A$) provided $\mathfrak{p}B_{\mathfrak{P}} = \mathfrak{P}B_{\mathfrak{P}}$ and the extension of residue fields $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$ is separable. The extension $A \longrightarrow B$ is said to be unramified if every $\mathfrak{P} \in SpecB$ is unramified over A.

Here is the connection with invariant subrings.

4.14 Proposition. *Keep the notation established above. The group G is small if and only if every height-one prime in S is unramified over R.*

The action of G on S fixing R induces a group homomorphism $G \longrightarrow \operatorname{End}_R(S)$. The funny multiplication rule on the twisted group ring is designed so that the extension

$$\delta \colon S\#G \longrightarrow \operatorname{End}_R(S),$$

which considers $s\sigma \in S\#G$ as the endomorphism $s\sigma$, is a ring homomorphism. In general, δ is neither injective nor surjective. When G is small, however, it is both.

4.15 Theorem (Auslander [Aus62, Prop. 3.4])**.** *Let $S = k[[x_1,\dots,x_n]]$ with k a field, and $G \subseteq \mathrm{GL}(n,k)$ a finite group acting linearly on S. Set $R = S^G$. If G is small, then the ring homomorphism $\delta \colon S\#G \longrightarrow \operatorname{End}_R(S)$ defined by*

$$\delta(s\sigma)(t) = s\sigma(t)$$

is an isomorphism.

We need a general lemma about normal domains due to Auslander and Buchsbaum [AB59], which will reappear repeatedly in later contexts.

4.16 Lemma. *Let A be a normal domain, M a finitely generated reflexive A-module, and N a finitely generated torsion-free A-module. A homomorphism $f \colon M \longrightarrow N$ is an isomorphism if and only if $f_{\mathfrak{p}}$ is an isomorphism for all primes $\mathfrak{p}$ of height one.*

Proof. We need only prove “if”. Set $K = \ker f$ and $C = \operatorname{cok} f$, so that we have the exact sequence

(4.16.1)
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & M \arrow[r, "f"] & N \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

Since $f_{(0)}$ is an isomorphism, $K_{(0)} = 0$, which means that K is annihilated by a nonzero element of A. But M is torsion-free, so $K = 0$. As for C, suppose that $C \neq 0$ and choose $\mathfrak{p} \in \operatorname{Ass} C$. Then $\mathfrak{p}$ has height at least 2. Localize 4.16.1 at $\mathfrak{p}$:

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0\,.$$

As M is reflexive, it satisfies (S_2), so $M_{\mathfrak{p}}$ has depth at least 2. On the other end, however, $C_{\mathfrak{p}}$ has depth 0, which contradicts the Depth Lemma. □

Proof of Theorem 4.15. Since $S\#G$ is isomorphic to a direct sum of copies of S as an S-module and an R-module, it is reflexive. Lemma 4.16 then assures us that it suffices to prove δ is an isomorphism in height one. At height one primes, the extension is unramified by Proposition 4.14, so we may assume for the proof that $R \longrightarrow S$ is unramified.

The multiplication map $\mu\colon S \otimes_R S \longrightarrow S$, defined by $\mu(s \otimes s') = ss'$, induces the short exact sequence

(4.16.2) $$0 \longrightarrow \mathscr{J} \longrightarrow S \otimes_R S \xrightarrow{\mu} S \longrightarrow 0\,,$$

where $\mathscr{J} = \ker \mu$ is generated by all elements of the form $s \otimes 1 - 1 \otimes s$ for $s \in S$. The extension $R \longrightarrow S$ being unramified is equivalent by **??** to (4.16.2) being a split exact sequence of $S \otimes_R S$-modules. Tensoring on the right with $S\#G$ thus gives another split exact sequence

(4.16.3) $$0 \longrightarrow \mathscr{J} \otimes_S S\#G \longrightarrow S \otimes_R S\#G \xrightarrow{\widetilde{\mu}} S\#G \longrightarrow 0\,.$$

Let $j\colon S \longrightarrow S \otimes_R S$ be a splitting for (4.16.2), and set $\epsilon = j(1)$. Then $\mu(\epsilon) = 1$ and

(4.16.4) $$(1 \otimes s - s \otimes 1)\epsilon = 0$$

for all $s \in S$. Considering $S \otimes_R S\#G$ as an $S \otimes_R S$-module via the diagonal, we claim that for an element $\sigma \in G$,

$$\widetilde{\mu}[(1 \otimes \sigma)(\epsilon)] = \begin{cases} 1 & \text{if } \sigma = \mathrm{id}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To see this, apply the endomorphism $1 \otimes \sigma \in \mathrm{End}_S(S \otimes_R S\#G)$ to both sides of (4.16.4), obtaining

$$(1 \otimes \sigma)[(1 \otimes s)\epsilon] = (1 \otimes \sigma)[(s \otimes 1)\epsilon],$$

which implies, since $\sigma\ s = \sigma(s)$, that

$$[1 \otimes \sigma(s)](1 \otimes \sigma)(\epsilon) = (1 \otimes \sigma)(\epsilon)[s \otimes 1]\,.$$

Collapsing all the tensor products with $\widetilde{\mu}$, and setting $\theta = \widetilde{\mu}((1 \otimes \sigma)(\epsilon))$, we see

$$\sigma(s)\theta = s\theta$$

for all $s \in S$. Hence either $\sigma = \mathrm{id}$ or $\theta = 0$, proving the claim.

Recall from the beginning of this section the ring homomorphism $\rho : S \longrightarrow S\#G$ defined by

$$\rho(s) = \sum_{\sigma \in G} \sigma(s)\ \sigma\,.$$

Applying δ to $\rho(s)$, for $s \in S$, gives an R-homomorphism of S, namely

$$\delta(\rho(s))(t) = \sum_{\sigma \in G} \sigma(s)\sigma(t) = \sum_{\sigma \in G} \sigma(st)\,. \tag{4.16.5}$$

This clearly satisfies $\delta(\rho(s))(t) \in R$ for all $t \in S$. As a special case, we see that $\rho(1)$ is the Reynolds operator.

Now we define a right splitting for $\delta\colon S\#G \longrightarrow \operatorname{End}_R(S)$. The ring homomorphism ρ induces a homomorphism

$$\omega\colon \operatorname{End}_R(S) \longrightarrow \operatorname{Hom}_S(S \otimes_R S, S \otimes_R S\#G)$$

defined by $\omega(f) = f \otimes \rho$. For any such f, one checks easily that $\omega(f)$ is S-linear with respect to the *right* action of S on the tensor products. From the target of ω, we have

$$\mathrm{ev}_\epsilon\colon \operatorname{Hom}_S(S \otimes_R S, S \otimes_R S\#G) \longrightarrow S \otimes_R S\#G\,,$$

which evaluates a homomorphism at the element $\epsilon \in S \otimes_R S$, and finally we have $\widetilde{\mu}\colon S \otimes_R S\#G \longrightarrow S\#G$, sending $t \otimes s\sigma$ to $ts\sigma$. We will show that

$$\widetilde{\mu}(\mathrm{ev}_\epsilon(\omega(f))) = f$$

for every $f \in \operatorname{End}_R(S)$. Apply the left-hand side to an element $s \in S$:

$$\begin{aligned}\widetilde{\mu}(\mathrm{ev}_\epsilon(\omega(f)))(s) &= \widetilde{\mu}(\mathrm{ev}_\epsilon(f \otimes \rho))(s)\\ &= \widetilde{\mu}((f \otimes \rho)(\epsilon))(s)\\ &= f(\widetilde{\mu}(1 \otimes \rho)(\epsilon)(s))\end{aligned}$$

since $\rho(\epsilon)(s) \in R$ by (4.16.5) above. This last is equal to $f(s)$ by the claim. Therefore $\delta\colon S\#G \longrightarrow \operatorname{End}_R(S)$ is a split surjection. Since both source and target of δ are R-modules of rank equal to $\operatorname{rank}_R(S)^2 = |G|^2$, δ is an isomorphism. $\square$

We now return to the functor $\mathscr{H}$ defined at the outset of this chapter. Recall that

$$\mathscr{H}\colon \operatorname{add}_{S\#G}(S\#G) \longrightarrow \operatorname{add}_R(S)$$

is the fixed-point functor $(-)^G$, which in particular satisfies

$$\mathscr{H}(S\#G) = (S\#G)^G = \rho(S) \cong S\,.$$

More precisely, any finitely generated projective $S\#G$-modules P is a free S-module by Proposition 4.6, so the fixed module P^G is, over R, a direct summand of a direct summand of copies of S.

4.17 Theorem. *Assume that G is small. Then the functor $\mathscr{H}$ from projective $S\#G$-modules to* $\operatorname{add}_R(S)$ *is an equivalence of categories.*

Proof. Considering S as an R-subalgebra of $S\#G$ via ρ, we claim that restriction of endomorphisms $\alpha\colon \operatorname{End}_{S\#G}(S\#G) \longrightarrow \operatorname{End}_R(\rho(S))$, that is, $\alpha(f) = f|_{\rho(S)}$, is an isomorphism. Since the projective $S\#G$-modules, resp. objects of $\operatorname{add}_R(S)$, correspond to the idempotents in these endomorphism rings, this will establish the Proposition.

Define a chain of R-algebra homomorphisms

$$S\#G \xrightarrow{\gamma} (S\#G)^* \xrightarrow{\beta} \operatorname{End}_{S\#G}(S\#G) \xrightarrow{\alpha} \operatorname{End}_R(\rho(S))$$

by

$$\gamma(s\sigma) = \sigma^{-1}(s)\sigma^{-1}$$
$$\beta(s\sigma)(t\tau) = (t\tau)(s\sigma)$$
$$\alpha(f) = f|_{\rho(S)}\,.$$

This composition sends $\tau \in G$ to the endomorphism

$$\sum_{\sigma\in G} \sigma(s)\,\sigma \qquad \mapsto \qquad \sum_{\sigma\in G} \sigma\tau(s)\,\sigma\,,$$

i.e. to τ considered as an endomorphism of $\rho(S)$. In other words, $\alpha\beta\gamma$ is the isomorphism δ of Theorem 4.15. Since γ and β are easily seen to be isomorphisms, this implies that α is as well. □

Putting together the one-one correspondences obtained so far, we have

4.18 Corollary. *Let k be a field, $S = k[[x_1,\dots,x_n]]$, and $G \subseteq \mathrm{GL}(n,k)$ a finite group acting linearly on S without pseudo-reflections and such that $|G|$ is invertible in k. Then there are one-one correspondences between*

- *the irreducible kG-modules;*
- *the indecomposable finitely generated projective $S\#G$-modules;*
- *the indecomposable finitely generated projective $\mathrm{End}_R(S)$-modules; and*
- *the indecomposable direct summands of S as an R-module.*

Explicitly, if $V_0,\dots,V_d$ are the non-isomorphic irreducible representations of G over k, then

$$M_j = (S \otimes_k V_j)^G, \qquad j = 0,\dots,d$$

are the indecomposable R-direct summands of S. They are in particular MCM R-modules. Furthermore, we have $\mathrm{rank}_R M_j = \dim_k V_j$.

§3 The McKay–Gabriel quiver

The one-one correspondence between projectives, representations, and certain MCM modules described in the previous section extends to an isomorphism of two graphs naturally associated to these data. We will meet a third graph in Chapter **??**.

We keep all the notation from 4.2, and additionally let $V_0, \dots, V_d$ be a complete set of the non-isomorphic irreducible k-representations of G, with V_0 the trivial representation k. The given linear action of G on S is induced from an n-dimensional representation of G on the space $V = \mathfrak{n}/\mathfrak{n}^2$ of linear forms.

4.19 Definition. The *McKay quiver* of $G \subseteq \mathrm{GL}(V)$ has

- vertices $V_0, \dots, V_d$, and
- m_{ij} arrows $V_i \longrightarrow V_j$ if the multiplicity of V_i in an irreducible decomposition of $V \otimes_k V_j$ is equal to m_{ij}.

NB: we're reversing the direction of the arrows in McKay's original formulation, but it's easy to see that similar reversals throughout will maintain all the isomorphisms below. We had to dualize somewhere.

In case k is algebraically closed of characteristic zero, the multiplicities m_{ij} in the McKay quiver can also be computed from the complex characters $\chi, \chi_0, \dots, \chi_d$ for $V, V_0, \dots, V_d$ [FH91, 2.10]:

$$m_{ij} = \langle \chi_i, \chi\chi_j \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_i(\sigma)\overline{\chi(\sigma)\chi_j(\sigma)}.$$

For each $i = 0, \dots, d$, we set $P_i = S \otimes_k V_i$, the corresponding indecomposable projective $S\#G$-module. Then in particular $P_0 = S \otimes_k V_0 = S$, and $\{P_0, \dots, P_d\}$ is a complete set of non-isomorphic indecomposable projective $S\#G$-modules by Prop. 4.9. The V_j are simple $S\#G$-modules via the surjection $S\#G \longrightarrow kG$, with minimal projective cover P_j. Since $\mathrm{pd}_{S\#G} V_j \leqslant n$ by Proposition 4.6, the minimal projective resolution of V_j over $S\#G$ thus has the form

$$0 \longrightarrow Q_n^{(j)} \longrightarrow Q_{n-1}^{(j)} \longrightarrow \cdots \longrightarrow Q_1^{(j)} \longrightarrow P_j \longrightarrow V_j \longrightarrow 0$$

with projective $S\#G$-modules $Q_i^{(j)}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, d$.

4.20 Definition. The *Gabriel quiver* of $G \subseteq \mathrm{GL}(V)$ has

- vertices $P_0, \ldots, P_d$, and
- m_{ij} arrows $P_i \longrightarrow P_j$ if the multiplicity of P_i in $Q_1^{(j)}$ is equal to m_{ij}.

4.21 Theorem ([Aus86b]). *The McKay quiver and the Gabriel quiver of R are isomorphic directed graphs.*

Proof. First consider the trivial module $V_0 = k$. The minimal $S\#G$-resolution of k was computed in Example 4.7; it is the Koszul complex

$$K_\bullet : \qquad 0 \longrightarrow S \otimes_k \bigwedge^n V \longrightarrow \cdots \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow 0\,.$$

To obtain the minimal $S\#G$-resolution of V_j, we simply tensor the Koszul complex with V_j over k, obtaining

$$0 \longrightarrow S \otimes_k \left(\bigwedge^n V \otimes_k V_j\right) \longrightarrow \cdots \longrightarrow S \otimes_k \left(V \otimes_k V_j\right) \longrightarrow S \otimes_k V_j \longrightarrow 0\,.$$

This displays $Q_1^{(j)} = S \otimes_k \left(V \otimes_k V_j\right)$, so that the multiplicity of P_i in $Q_1^{(j)}$ is equal to that of V_i in $V \otimes_k V_j$. $\square$

4.22 Example. Take $n = 3$, and write $S = k[\![x, y, z]\!]$. Let $G = \mathbb{Z}/2\mathbb{Z}$, with the generator acting on $V = kx \oplus ky \oplus kz$ by negating each variable. Then $R = S^G = k[\![x^2, xy, xz, y^2, yz, z^2]\!]$. There are only two irreducible representations of G, namely the trivial representation k and its negative, which is isomorphic to the inverse determinant representation $V_1 = \det(V)^{-1} = \bigwedge^3 V^*$. The Koszul complex

$$0 \longrightarrow S \otimes \bigwedge^3 V \longrightarrow S \otimes_k \bigwedge^2 V \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow 0$$

resolves k, while the tensor product

$$\begin{tikzcd}
0 \arrow[r] & S \otimes \left(\bigwedge^3 V \otimes_k \bigwedge^3 V^*\right) \arrow[r] & S \otimes_k \left(\bigwedge^2 V \otimes_k \bigwedge^3 V^*\right) \arrow[r] & {} \\
S \otimes_k \left(V \otimes_k \bigwedge^3 V^*\right) \arrow[r] & S \otimes_k \bigwedge^3 V^* \arrow[r] & 0
\end{tikzcd}$$

is canonically isomorphic to

$$0 \longrightarrow S \longrightarrow S \otimes_k V^* \longrightarrow S \otimes_k \bigwedge^2 V^* \longrightarrow S \otimes_k \bigwedge^3 V^* \longrightarrow 0\,.$$

Since the given representation is just 3 copies of $\bigwedge^3 V^*$, we obtain the McKay quiver

$$\begin{tikzcd}
V_0 \arrow[r, bend left=30] \arrow[r, bend left=20] \arrow[r, bend left=10] & V_1 \arrow[l, bend left=10] \arrow[l, bend left=20] \arrow[l, bend left=30]
\end{tikzcd}$$

or the Gabriel quiver

$$\begin{tikzcd}
S \otimes_k V_0 \arrow[r, bend left=30] \arrow[r, bend left=20] \arrow[r, bend left=10] & S \otimes_k V_1 \arrow[l, bend left=10] \arrow[l, bend left=20] \arrow[l, bend left=30]
\end{tikzcd}\ .$$

Taking fixed points as specified in Corollary 4.18, we find MCM modules

$$M_0 \cong R \qquad \text{and} \qquad M_1 = (S \otimes_k V_1)^G\,.$$

Since V_1 is the negative of the trivial representation, the fixed points of $S \otimes_k V_1$ are generated over R by the linear forms of S, so that M_1 is the submodule of S generated by (x, y, z). This is isomorphic to the ideal (x^2, xy, xz) of R. In particular we recover the obvious R-direct sum decomposition $S = R \oplus R(x, y, z)$ of S.

From now on, we draw the McKay quiver for a group G, and refer to it as the McKay–Gabriel quiver.

4.23 Example. Let $n = 2$ now, and write $S = k[\![u,v]\!]$. Let $r \geqslant 2$ be an integer not divisible by char(k), and choose $0 < q < r$ with $(q,r) = 1$. Take $G = \langle g \rangle \cong \mathbb{Z}/r\mathbb{Z}$ to be the cyclic group of order r generated by

$$g = \begin{pmatrix} \zeta_r & \\ & \zeta_r^q \end{pmatrix} \qquad \in \mathrm{GL}(2,k),$$

where ζ_r is a primitive r^{th} root of unity. Let $R = k[\![u,v]\!]^G$ be the corresponding ring of invariants.

As G is Abelian, it has exactly r irreducible representations, each of which is one-dimensional. We label them $V_0, \dots, V_{r-1}$, where the generator g is sent to ζ_r^i in V_i. The given representation V of G is isomorphic to $V_1 \oplus V_q$, so that for any j we have

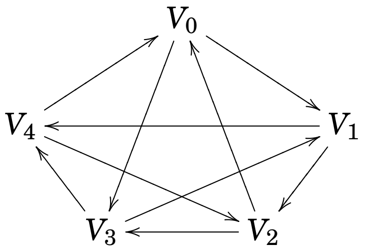
$$V \otimes_k V_j \cong V_{j+1} \oplus V_{j+q}\,,$$

where the indices are of course to be taken modulo r. The corresponding MCM R-modules are $M_j = (S \otimes_k V_j)^G$, each of which is an R-submodule of S:

$$M_j = R\left(u^a v^b \;\mid\; j + a + qb \equiv 0 \mod r\right).$$

The general picture is a bit chaotic, so here are a few particular examples.

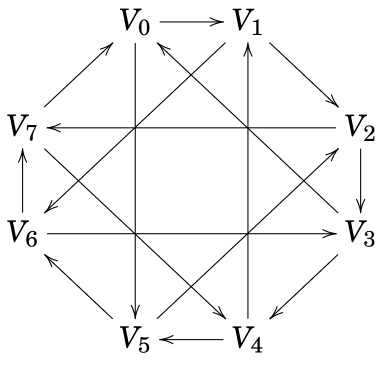
Take $r = 5$ and $q = 3$. Then $R = k[\![u^5, u^2v, uv^3, v^5]\!]$. The McKay–Gabriel quiver takes the following shape.



The associated indecomposable MCM R-modules appearing as R-direct summands of S are the ideals

$$
\begin{aligned}
M_0 &= R \\
M_1 &= R(u^4, uv, v^3) \cong (u^5, u^2v, uv^3) \\
M_2 &= R(u^3, v) \cong (u^5, u^2v) \\
M_3 &= R(u^2, uv^2, v^4) \cong (u^5, u^4v^2, u^3v^4) \\
M_4 &= R(u, v^2) \cong (u^5, u^4v^2).
\end{aligned}
$$

For another example, take $r = 8, q = 5$, so that $R = k[\![u^8, u^3v, uv^3, v^8]\!]$. The McKay–Gabriel quiver looks like



and the indecomposable MCM R-modules arising as direct summands of S

are

$$\begin{aligned}
M_0 &= R \\
M_1 &= R(u^7, u^2v, v^3) \cong (u^8, u^3v, uv^3) \\
M_2 &= R(u^6, uv, v^6) \cong (u^8, u^3v, u^2v^6) \\
M_3 &= R(u^5, v) \cong (u^8, u^3v) \\
M_4 &= R(u^4, u^2v^2, v^4) \cong (u^8, u^6v^2, u^4v^4) \\
M_5 &= R(u^3, uv^2, v^7) \cong (u^8, u^6v^2, u^5v^7) \\
M_6 &= R(u^2, u^5v, v^2) \cong (u^2v^6, u^5v^7, v^8) \\
M_7 &= R(u, v^5) \cong (uv^3, v^8).
\end{aligned}$$

Finally, take $r = n+1$ arbitrary, and $q = n$. Then $R = k\llbracket u^{n+1}, uv, v^{n+1}\rrbracket \cong k\llbracket x, y, z\rrbracket/(xz - y^{n+1})$ is isomorphic to an (A_n) hypersurface singularity (see the next chapter). There are $n+1$ irreducible representations $V_0, \dots, V_n$, and the McKay–Gabriel quiver looks like the one below.

$$\begin{tikzcd}
& & V_0 \arrow[dll, shift left] \arrow[drr, shift left] & & \\
V_1 \arrow[urr, shift left] \arrow[r, shift left] & V_2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & V_{n-1} \arrow[l, shift left] \arrow[r, shift left] & V_n \arrow[l, shift left] \arrow[ull, shift left]
\end{tikzcd}$$

The non-free indecomposable MCM R-modules take the form

$$M_j = R(u^a v^b \mid b - a \equiv j \mod n+1)$$

for $j = 1, \dots, n$. They have presentation matrices over $k\llbracket u^{n+1}uv, v^{n+1}\rrbracket$

$$\varphi_j = \begin{pmatrix} (uv)^{n+1-j} & -u^{n+1} \\ -v^{n+1} & (uv)^j \end{pmatrix}$$

or over $k\llbracket x, y, z\rrbracket$

$$\varphi_j = \begin{pmatrix} y^{n+1-j} & -x \\ -z & y^j \end{pmatrix}.$$

§4 Exercises

4.24 Exercise. In the notation of 4.2, prove that $\mathfrak{m} = \mathfrak{n} \cap R$ is the unique maximal ideal of R.

4.25 Exercise. Prove that R is Noetherian, as follows. (Notice that $|G|$ being invertible in k is essential for this! Nagata's counterexample to Hilbert 14 says so. And Fogarty gave an example of a finite group G acting on a CM ring S such that S^G is Noetherian but not CM. [PAMS 1981])

1. Show $IS \cap R = I$ for every ideal I of R.
2. Deduce that R has ACC on ideals.

4.26 Exercise. Prove R is normal:

1. $Q(R) = Q(S)^G$
2. If $x \in \overline{R}$, $x^n + c_1 x^{n-1} + \cdots + c_n = 0$, then $x \in S \cap Q(R)$, so $R = \overline{R}$.

4.27 Exercise. Use the converse of NAK for complete rings (an exercise in Eisenbud) to prove that S is a finitely generated R-module.

4.28 Exercise. Prove R is complete (Cauchy sequences: if x_i is Cauchy, it converges in S, to, say, s, and then $\sigma(s) - s$ is zero mod all powers of $\mathfrak{n}$, for all $\sigma \in G$; finish with the Krull Intersection Theorem.

4.29 Exercise. Let A be a local ring and M, n two finitely generated A-modules. Then $\operatorname{depth} \operatorname{Hom}_A(M,N) \geqslant \min\{2, \operatorname{depth} N\}$.