MIXED CHARACTERISTIC HYPERSURFACES OF FINITE COHEN–MACAULAY TYPE

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Abstract. We define the mixed ADE singularities, which are generalizations of the ADE plane curve singularities to the case of mixed characteristic. The ADE plane curve singularities are precisely the equicharacteristic plane curve singularities of finite Cohen–Macaulay type; we show that the mixed ADE singularities also have finite Cohen–Macaulay type.

Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of Krull dimension d. A non-zero R-module M is *maximal Cohen–Macaulay* (MCM) provided it is finitely generated and there exists an M-regular sequence $x_1, \dots, x_d$ in the maximal ideal $\mathfrak{m}$. In particular, R is a *Cohen–Macaulay* (CM) ring if R is a MCM module over itself. The ring R is said to have *finite Cohen–Macaulay type* (or finite CM type) if there are, up to isomorphism, only finitely many indecomposable MCM R-modules.

The complete equicharacteristic hypersurface singularities of finite CM type have been completely characterized ([2], [4], [5], [8], [12]). A complete equicharacteristic hypersurface singularity is a ring of the form $R = A/(f)$, where $A = k[[x_0, \dots, x_d]]$ is the ring of formal power series over an algebraically closed field k and f is a nonzero element in the square of the maximal ideal of A. For $d \geq 1$ and $\operatorname{char}(k) \neq 2$ it is known that such a singularity has finite CM type if and only if $R \cong k[[x_0, \dots, x_d]]/(g + x_2^2 + \cdots + x_d^2)$, where $g \in k[x_0, x_1]$ defines a simple plane curve singularity. For $\operatorname{char}(k) \neq 2, 3, 5$, these simple plane curve singularities are defined by the following polynomials, corresponding to certain Dynkin diagrams:

$$\begin{array}{lll} (\mathrm{A}_n) & x_0^2 + x_1^{n+1}, & (n \geq 1); \\ (\mathrm{D}_n) & x_1(x_0^2 + x_1^{n-2}), & (n \geq 4); \\ (\mathrm{E}_6) & x_0^3 + x_1^4; & \\ (\mathrm{E}_7) & x_0(x_0^2 + x_1^3); & \\ (\mathrm{E}_8) & x_0^3 + x_1^5. & \end{array}$$

When $\operatorname{char}(k)$ is one of $2, 3$, or 5, there are some additional normal forms [13].

This paper is concerned with showing certain examples of complete one-dimensional hypersurfaces of mixed characteristic have finite CM type. Recall that a complete one-dimensional hypersurface of mixed characteristic has the form $R = V[[y]]/(f(y))$, where (V, pV) is a discrete valuation ring and p is a prime number. These examples, which we call the mixed ADE singularities, are the natural extensions of the simple plane curve singularities to the situation where R does not contain a field. See Definition 3.1. Since Herzog has shown [7] that a complete Gorenstein local ring of finite CM type must be a hypersurface, these examples represent a step toward a full classification of the one-dimensional Gorenstein local rings of finite CM type.

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To prove that a given ring has finite CM type, we compute the Auslander–Reiten quiver. The AR quiver encapsulates much of the structure of the category of MCM modules over the ring. This structure is given in terms of Auslander–Reiten sequences, also known as almost split sequences. Section 1 reviews the relevant properties of AR sequences and quivers.

When computing AR quivers, we need a way to know whether we have the complete quiver. Section 2 is devoted to proving such a result in this context (Theorem 2.9). This type of result is called a Brauer–Thrall type theorem in [15]. All existing versions of Brauer–Thrall type theorems assume that ring in question contains a field; we replace this restriction by the much milder assumption that the residual characteristic p not be a zerodivisor.

The third section consists of the definitions of the mixed ADE singularities and computation of their AR quivers. Not surprisingly, the AR quivers for the mixed ADE singularities closely resemble their equicharacteristic counterparts. The assumption of mixed characteristic does, however, impose some technical difficulties. For instance, in two of the mixed ADE singularities, (D_n) and (E_7'), the residual characteristic p is a zerodivisor. This prevents us from applying Theorem 2.9. We resort to verifying conditions due to J.A. Drozd and A.V. Roĭter (Theorem 3.3) to prove that these rings have finite CM type. While effective, this approach does not allow us to enumerate all the MCM modules as we can in the other cases.

1. Auslander–Reiten quivers

For this section, R is a complete CM local ring with algebraically closed residue field. The theory of Auslander–Reiten quivers, originally developed by M. Auslander and I. Reiten for the representation theory of Artin algebras, has been extended to the representation theory of maximal Cohen–Macaulay R-modules. In this context, they turn out to have an intimate relationship with the singularity defined by R. They also give a wealth of information about the structure of the category of MCM R-modules. Most of the results cited here can be found in Yoshino's excellent monograph [16].

Let M be an indecomposable MCM R-module. An *Auslander–Reiten (AR) sequence ending in* M is a nonsplit short exact sequence

$$0 \longrightarrow N \xrightarrow{\ p\ } E \xrightarrow{\ q\ } M \longrightarrow 0 \tag{1}$$

such that (a) N is an indecomposable MCM R-module, and (b) any homomorphism of MCM R-modules $L \longrightarrow M$ that is not a split surjection factors through q. AR sequences are unique up to isomorphism of exact sequences when they exist. We say also that (1) is an *AR sequence starting from* N. The ring R is said to *admit AR sequences* if for every nonfree indecomposable MCM R-module M there is an AR sequence ending in M. A significant result of M. Auslander gives a necessary and sufficient condition for the existence of AR sequences.

Theorem 1.1 ([1]). *The ring R admits AR sequences if and only if R is an isolated singularity (that is, each localization $R_{\mathfrak{p}}$ with $\mathfrak{p} \neq \mathfrak{m}$ is a regular local ring).*

In the AR sequence (1), N is called the Auslander translation of M, and we write $N = \tau(M)$. The Auslander translation of M is easily calculated. Recall that the Auslander transpose $\mathrm{tr}(M)$ of M is $\mathrm{Coker}\,\Phi^*$, where Φ is a presentation matrix for M.

Lemma 1.2 ([16, 3.13]). *Let $d = \dim(R)$. Then $\tau(M) \cong \mathrm{Hom}_R(\mathrm{syz}^d_R \,\mathrm{tr}(M), \omega)$, where ω is the canonical module for R.*

It follows that $\tau(\tau(M)) \cong M$ [16, 2.14].

Closely related to AR sequences are irreducible homomorphisms. A homomorphism of MCM R-modules $\varphi : M \longrightarrow N$ is *irreducible* provided (a) φ is neither a split injection nor a split surjection, and (b) if φ factors through a MCM module X

$$\begin{array}{ccc} M & \xrightarrow{\ \varphi\ } & N \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & X & \end{array} \tag{2}$$

then either α is a split injection or β is a split surjection.

The next lemma follows from [16, 2.12] and the proof there.

Lemma 1.3 ([16, 2.12])**.** *Let M and L be indecomposable MCM R-modules, and assume that there exists an AR sequence (1) ending in M. The following conditions are equivalent.*

1. *L is isomorphic to a direct summand of E.*
2. *There is an irreducible homomorphism $L \longrightarrow M$.*

Each of these implies that the composition $L \longrightarrow E \longrightarrow M$ is an irreducible homomorphism.

In fact, the irreducible homomorphisms $L \longrightarrow M$ form a finite-dimensional k-vector space, and [16, 5.5] the dimension of this vector space is equal to the number of copies of L in the direct-sum decomposition of E.

We encode all the above data into a graph, called the Auslander–Reiten quiver of R.

Definition 1.4. *Assume R is an isolated singularity. The AR quiver Γ of R is a graph consisting of vertices, arrows, and dotted lines. The vertices are the isomorphism classes of indecomposable MCM R-modules. We draw n arrows $[A] \longrightarrow [B]$ if and only the dimension of the k-vector space of irreducible homomorphisms $A \longrightarrow B$ is n. We draw a dotted line between $[A]$ and $[B]$ if $A \cong \tau(B)$.*

The following lemma will be essential to understanding the structure of the AR quiver.

Lemma 1.5 ([16, 5.9])**.** *Assume R (complete with algebraically closed residue field) is an isolated singularity. Then the AR quiver Γ of R is a locally finite graph (that is, each vertex of Γ has only finitely many arrows starting from it or ending in it).*

2. A Brauer–Thrall Theorem in Mixed Characteristic

In proving that certain complete equicharacteristic hypersurfaces have finite Cohen–Macaulay type, Y. Yoshino (like R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer [2]) uses the following theorem [16, 6.2]:

Theorem 2.1. *Let R be a complete equicharacteristic CM local ring. Assume that R is an isolated singularity and that the residue field is algebraically closed. Let Γ be the Auslander–Reiten quiver of R, and assume that Γ^0 is a connected component of Γ with bounded multiplicity type. Then $\Gamma^0 = \Gamma$ and Γ is a finite graph. In particular, R has finite CM type.*

In the statement of Theorem 2.1, to say that Γ^0 has *bounded multiplicity type* means that there exists an integer a such that for any $[M]$ in Γ^0, $e(M) < a$. Recall that the multiplicity $e(M)$ of a module M over a local ring R is the d^{th} derivative of the Hilbert polynomial of M, where

$d = \dim(R)$. If R is an integral domain (or, more generally, if M is free of constant rank at the associated primes of R), then $e(M) = e(R) \cdot \operatorname{rank}(R)$.

All issues of connectedness in Γ refer to the undirected graph obtained by replacing each arrow by an undirected edge and ignoring the dotted lines.

Theorem 2.1 is called a Brauer–Thrall type theorem in [15], by analogy with the First Brauer–Thrall Theorem in the representation theory of Artin algebras. We will use a result of this form to help us classify the mixed characteristic hypersurfaces of finite CM type. See Theorem 2.9.

• The following notations will be in effect for the rest of this section. Let V be a complete discrete valuation ring of characteristic zero with uniformizing parameter π and algebraically closed residue field of characteristic $p > 0$. There will be certain restrictions on p in what follows. Let R be a one-dimensional hypersurface over V, that is, $R = V[[y]]/(f(y))$ for some non-zero power series f in the maximal ideal of $V[[y]]$. Denote the maximal ideal of R by $\mathfrak{m}$. We assume that π is not a factor of f, that is, π is a nonzerodivisor in R.

Note first that we may assume that f is a monic polynomial in y with coefficients in V. Write $f = \sum_{n=0}^{\infty} u_n \pi^{a_n} y^n$, where the a_n are nonnegative integers and u_n are units of V. Since π does not divide f by assumption, $a_n = 0$ for some $n > 0$. Let m be the smallest integer such that $a_m = 0$. Then f is *regular of order* m (see [9, IV]). By the Weierstrass Preparation Theorem ([9, IV, 9.2]), there is a linear change of variable, σ, such that $R \cong V[[y]]/(\sigma(f))$ and $\sigma(f)$ is a monic polynomial of degree m in which the coefficient of y^i is divisible by π for each $i < m$.

It follows from [9, IV, 9.1] that R is a finitely generated V-module, generated by $\{1, y, \dots, y^{m-1}\}$.

Recall that the *Noether different* $N_V(R)$ of R over V is defined as follows: let $\mu : R \otimes_V R \longrightarrow R$ be the multiplication map, and let J be the kernel, so we have the exact sequence

$$0 \longrightarrow J \longrightarrow R \otimes_V R \xrightarrow{\mu} R \longrightarrow 0.$$

Set $N_V(R) = \mu(\operatorname{Ann}_{R\otimes_V R}(J))$.

Our interest in the Noether different $N_V(R)$ stems from the fact that reduction modulo a nonzerodivisor x contained in $N_V(R)$ induces an embedding of the category of MCM R-modules into the category of $R/(x)$-modules. Such an element x is called an *efficient parameter* by Yoshino [16]. The embedding will preserve indecomposability and multiplicity, and will allow us to apply a lemma due to Harada–Sai [6] to prove a version of the Brauer–Thrall theorem.

The key fact about the Noether different is the following from [11, Section 11.5].

Lemma 2.2. *Let V and R be as above, and let M be an $R \otimes_V R$-module. Then $N_V(R)$ annihilates the Hochschild cohomology $H^i_V(R, M)$ for all $i > 0$.*

En route to identifying an efficient parameter, the following easy lemma will be useful.

Lemma 2.3. *With notation as above, $f'(y) \in N_V(R)$.*

Lemma 2.4. *Let V and R be as above, and further assume that R is reduced. Then there is an integer t such that $\pi^t \in N_V(R)$.*

Proof. Let K denote the quotient field of V. Then $R[\pi^{-1}] = K[y]/(f(y))$ is a finite-dimensional K-algebra. It is easy to see that, since f is irreducible over V, f is also irreducible over K. So $K[y]/(f(y))$ is a simple field extension of K. Since K has characteristic zero, the extension is separable. Thus there exist polynomials g and h in $K[y]$ such that $gf + f'h = 1$. Clearing denominators and killing f, we see that, for some t, π^t is in the ideal of R generated by $f'(y)$. ☐

The proofs of the following two statements are identical to those in [16]

Proposition 2.5. *Let V and R be as above, and let M and N be MCM R-modules. Assume that $\pi^t \in N_V(R)$ for some $t \geq 1$. Then for any homomorphism $\varphi : M/\pi^{2t}M \longrightarrow N/\pi^{2t}N$, there exists a homomorphism $\psi : M \longrightarrow N$ such that $\varphi \otimes_R R/(\pi^t) = \psi \otimes_R R/(\pi^t)$.*

Corollary 2.6. *Let V and R be as above, and assume that $\pi^t \in N_V(R)$ for some $t \geq 1$. Let M be a MCM R-module. Then M is indecomposable if and only if $M/\pi^{2t}M$ is indecomposable.*

• For the rest of this section, we fix an integer t such that $\pi^t \in N_V(R)$. Further assume that R is an isolated singularity (*i.e.*, reduced), so that R admits AR sequences. Let Γ be the AR quiver for R, and let Γ^0 be a connected component. Assume that Γ^0 has bounded multiplicity type, that is, there exists an integer a such that $e(M) \leq a$ for any vertex $[M]$ in Γ^0. Then for any such M, the length $\ell(M/\pi^{2t}M)$ is bounded above by ab, where b is the smallest integer such that $(\pi, y)^b \subseteq \pi^{2t}R$ [16, 1.7].

In what follows, call a homomorphism φ between two R-modules *trivial modulo* π^{2t} if $\varphi \otimes_R R/(\pi^{2t}) = 0$. The next result is referred to as a Harada–Sai Lemma in [15]. The original Harada–Sai Lemma is as follows [6]: Let S be a Artinian ring and fix a nonnegative integer r. Let N_i, $0 \leq i \leq 2^r$, be indecomposable nonzero finitely generated S-modules such that $\ell(N_i) \leq r$ for $i = 0, \dots, 2^r$, and let $g_i : N_{i-1} \longrightarrow N_i$, $i = 1, \dots, 2^r$, be homomorphisms which are not isomorphisms. Then the composition $g_{2^r} g_{2^r-1} \cdots g_2 g_1$ is zero.

Lemma 2.7. [15, 6.20] *Keep the notation introduced thus far, and fix an integer $r \geq 0$. Let M_i, $0 \leq i \leq 2^r$, be indecomposable MCM R-modules, and let $f_i : M_{i-1} \longrightarrow M_i$, $i = 1, \dots, 2^r$, be homomorphisms which are not isomorphisms. Assume that $\ell(M_i/\pi^{2t}M_i) \leq r$ for $i = 0, \dots, 2^r$. Then the composition $f_{2^r} f_{2^r-1} \cdots f_2 f_1$ is trivial modulo π^{2t}.*

Proof. In order to apply the original Harada–Sai Lemma to $S = R/(\pi^{2t})$, $N_i = M_i/\pi^{2t}M_i$, and $g_i = f_i \otimes_R S$, we need only show that $M_i/\pi^{2t}M_i$ is indecomposable for $i = 0, \dots, 2^r$, and that no $f_i \otimes_R S$ is an isomorphism. The first statement follows from Corollary 2.6. For the second, assume that $f_i \otimes_R S$ is an isomorphism for some i. Then by [3, 21.13], f_i is an isomorphism, a contradiction. □

Lemma 2.8. *Keep the notation introduced thus far. Let M and N be two indecomposable MCM R-modules, and let $\varphi : M \longrightarrow N$ be a homomorphism that is not trivial modulo π^{2t}. Then $[M] \in \Gamma^0$ iff $[N] \in \Gamma^0$. Moreover, if either is in Γ^0, then $[M]$ and $[N]$ are connected by a path in Γ^0 of length less than 2^{ab}.*

Proof. First assume that $[N]$ is in Γ^0. Fix a non-negative integer n, to be determined later. Assume there is no path Π in the *undirected* graph Γ such that (1) Π connects $[M]$ to $[N]$ and (2) Π has length strictly less than n. We claim that there is a chain of homomorphisms between indecomposable MCM R-modules

$$M \xrightarrow{g} N_n \xrightarrow{f_n} N_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N \tag{3}$$

such that each f_i is irreducible and the composition $f_1 f_2 \cdots f_n g$ is not trivial modulo π^{2t}.

We construct the chain (3) by induction on n. If $n = 0$, then we take $g = \varphi$, so there is nothing to show. Assume $n \geq 1$. By the induction hypothesis, there is a chain

$$M \xrightarrow{g} N_{n-1} \xrightarrow{f_{n-1}} N_{n-2} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N$$

such that each f_i, $i = 1, \cdots, n-1$ is irreducible, each N_i is indecomposable, and the composition $f_1 f_2 \cdots f_{n-1} g$ is not trivial modulo π^{2t}. The assumption that there is no chain in the AR quiver of length less than n implies that g is not an isomorphism. We will extend this chain.

First suppose that N_{n-1} is not free. Then there is an AR sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{q} N_{n-1} \longrightarrow 0 \tag{4}$$

ending in N_{n-1}. Write E as a direct sum of indecomposable MCM R-modules, $E = \bigoplus_{i=1}^{s} E_i$. Then we can decompose $q = \sum_{i=1}^{s} q_i$, where each $q_i : E_i \longrightarrow N_{n-1}$ is an irreducible homomorphism by Lemma 1.3. The homomorphism $g : M \longrightarrow N_{n-1}$ is not an isomorphism, so is not a split injection since both modules are indecomposable. The defining property of the AR sequence ending in N_{n-1} then implies the existence of a homomorphism $h : M \longrightarrow E$ such that the triangle

$$\begin{array}{ccc} E & \xrightarrow{\quad q \quad} & N_{n-1} \\ {\scriptstyle h}\nwarrow & & \nearrow{\scriptstyle g} \\ & M & \end{array}$$

commutes. Write $h = \sum_{i=1}^{s} h_i$ for homomorphisms $h_i : M \longrightarrow E_i$. Since (4) is not split, no h_i is an isomorphism. Since $f_1 f_2 \cdots f_{n-1} g$ is not trivial modulo π^{2t}, $f_1 f_2 \cdots f_{n-1}(qh)$ is not trivial modulo π^{2t}. Then $f_1 f_2 \cdots f_{n-1}(q_j h_j)$ is not trivial modulo π^{2t} for some j, $1 \le j \le s$, and we have the chain of homomorphisms

$$M \xrightarrow{h_j} E_j \xrightarrow{q_j} N_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N$$

such that p_j is an irreducible homomorphism between indecomposable MCM modules, h_j is not an isomorphism, and the composition is not trivial modulo π^{2t}. We have extended the chain and completed the proof of the claim in the case where N_{n-1} is not free.

Now suppose that $N_{n-1} \cong R$ is free. Then since g is not an isomorphism, $g(M) \subseteq \mathfrak{m}$, the maximal ideal of R. Since $\dim(R) = 1$, $\mathfrak{m}$ is a MCM R-module, and we have

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & R \\ {\scriptstyle g'}\searrow & & \nearrow{\scriptstyle h} \\ & \mathfrak{m} & \end{array}$$

where h is the natural inclusion. We claim that h is irreducible. Suppose there is a factorization

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\quad h \quad} & R \\ {\scriptstyle \alpha}\searrow & & \nearrow{\scriptstyle \beta} \\ & X & \end{array}$$

with X a MCM R-module. If β is not a split surjection, then $\beta(x) = x$ for each $x \in \mathfrak{m}$, and $\beta\alpha(\mathfrak{m}) = \mathfrak{m}$, so α is a split monomorphism. This shows that the inclusion $h : \mathfrak{m} \longrightarrow R$ is irreducible. Decompose $\mathfrak{m} = \bigoplus_{i=1}^{s} E_i$ with each E_i indecomposable, and write $g' = \sum_{i=1}^{s} g'_i$, $h = \sum_{i=1}^{s} h_i$ for maps $g'_i : M \longrightarrow E_i$ and $h_i : E_i \longrightarrow R$. Then, as before, $f_1 f_2 \cdots f_{n-1} h_j g'_j$ is nontrivial modulo π^{2t} for some j, and each h_j is irreducible. This extends the chain (2) and completes the proof of the claim.

Suppose now that $[M] \notin \Gamma^0$. Put $n = 2^{ab}$. Since there is no path Π in Γ of length less than n that connects $[N]$ to $[M]$, we obtain the chain of homomorphisms (3). Since $f_1 f_2 \cdots f_n g$ is non-trivial mod π^{2t}, so is $f_1 f_2 \cdots f_n$, and we have a contradiction to Lemma 2.7.

Suppose, conversely, that $[M] \in \Gamma^0$. We use an argument exactly dual to the one above to prove that $[N] \in \Gamma^0$: The claim this time is that if there is no path of length less than n connecting $[M]$ to $[N]$, then there is chain of homomorphisms between indecomposable MCM R-modules

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{g} N$$

such that each f_i is an irreducible homomorphism, g is not an isomorphism, and the composition is not trivial modulo π^{2t}. □

Theorem 2.9. *Let (V, π) be a complete discrete valuation ring with algebraically closed residue field, and set $R = V[[y]]/(f)$ for some non-zero non-unit $f \in V[[y]]$. Assume that R is reduced and that π is not a factor of f. Let Γ be the Auslander–Reiten quiver of R, and let Γ^0 be a nonempty connected component of Γ with bounded multiplicity type. Then $\Gamma^0 = \Gamma$ and Γ is a finite graph. In particular, R has finite CM type.*

Proof. Let $M \in \Gamma^0$. Then, by Nakayama's lemma, there exists an element $x \in M \setminus \pi^{2t}M$, so there is a homomorphism $\varphi : R \longrightarrow M$, taking 1 to x, such that φ is nontrivial modulo π^{2t}. Lemma 2.8 shows that $[R] \in \Gamma^0$. Then for any $[N] \in \Gamma$ we can define a homomorphism $\psi : R \longrightarrow N$ in the same way and deduce that $[N] \in \Gamma^0$.

To see that Γ is a finite graph, note that by Lemma 2.8, any vertex of Γ is connected to $[R]$ by a chain of arrows of length less than 2^{ab}. Since Γ is a locally finite graph (Lemma 1.5), Γ is finite. □

3. Mixed ADE Singularities

The goal of this section is to compute the Auslander–Reiten quivers of the mixed ADE singularities, and thereby show that they have finite CM type. The mixed ADE singularities are the natural generalizations of the simple plane curve singularities over a field, which are known to be precisely those plane curve singularities of finite CM type (see the introduction). All our proofs in this section are modeled on those in [15] and [16].

Definitions and Preliminaries. Throughout this subsection, we keep the notation of Section 2: Let (V, π) be a complete discrete valuation ring of characteristic zero and residual characteristic $p > 0$. Let $R = V[[y]]/(f)$ be a hypersurface over V, where f is a non-zero non-unit of $S = V[[y]]$. We always assume that f is square-free, that is, R is an isolated singularity.

Definition 3.1. *We say that R is a* mixed ADE singularity *if R is isomorphic to one of the following.*

(A_n)	$y^2 + \pi^{n+1}$	$(n \geq 2)$	(A'_n)	$\pi^2 + y^{n+1}$	$(n \geq 2)$
(D_n)	$\pi(y^2 + \pi^{n-2})$	$(n \geq 4)$	(D'_n)	$y(\pi^2 + y^{n-2})$	$(n \geq 4)$
(E_6)	$y^3 + \pi^4$		(E'_6)	$\pi^3 + y^4$	
(E_7)	$y(y^2 + \pi^3)$		(E'_7)	$\pi(\pi^2 + y^3)$	
(E_8)	$y^3 + \pi^5$		(E'_8)	$\pi^3 + y^5$	

In order to apply Theorem 2.9 to conclude that the Auslander–Reiten quivers of these rings are connected, we need to know that π is not a zerodivisor. This rules out the equations (D_n) and (E_7'); we will deal with these separately.

We now briefly review some relevant facts about Auslander–Reiten quivers in the specific context of this section.

Lemma 3.2. *The AR translation of a nonfree indecomposable MCM R-module M is given by $\tau(M) \cong \mathrm{syz}^1_R(M)$.*

Proof. In general (Lemma 1.2), $\tau(M) \cong (\mathrm{syz}^d_R \operatorname{tr}(M))'$, where $d = \dim(R)$, tr is the Auslander transpose of M, and $(-)'$ means the canonical dual $\mathrm{Hom}_R(-, \omega_R)$. By definition of the Auslander translation, dualizing a minimal free presentation $F_1 \to F_0 \to M \to 0$ of M into R gives

$$0 \longrightarrow M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \operatorname{tr}(M) \longrightarrow 0. \tag{5}$$

Since the original resolution was minimal, (5) is as well and therefore is a minimal free resolution of $\operatorname{tr}(M)$. The first syzygy in this sequence is thus $\mathrm{syz}^1_R(\operatorname{tr}(M))$, which, since R is Gorenstein, is isomorphic to $\tau(M)^*$. Dualizing again gives $0 \to \tau(M) \to F_0 \to M \to 0$, so $\tau(M) = \mathrm{syz}^1_R(M)$. □

We also use the following two more general facts. The first tells us how to identify the AR sequence ending in a given module if we run into it on the road, and the second gives a map to find it.

Let M be a nonfree indecomposable MCM R-module. The AR sequence ending in M can be represented by an element of $\mathrm{Ext}^1_R(M, \tau(M))$. Since M is locally free on the punctured spectrum of R (Lemma 1.1), $\mathrm{Ext}^1_R(M, \tau(M))$ has finite length. In fact, the proof of [16, 3.11] shows that the socle of $\mathrm{Ext}^1_R(M, \tau(M))$ is a one-dimensional vector space over the residue field of R. Choose a generator s for this socle. Then [16, 3.11] s represents the AR sequence ending in M.

The second fact deals with the theory of matrix factorizations over the hypersurface $R = S/(f)$. We refer to [16, Chapter 7] for the details.

Let M be a MCM R-module with no nonzero free summands. There is an exact sequence of S-modules

$$0 \longrightarrow S^m \xrightarrow{\varphi} S^m \longrightarrow M \longrightarrow 0$$

where m is the number of generators required for M. We can regard φ as an $m \times m$ matrix with entries in the maximal ideal of S. There is another $m \times m$ matrix ψ such that both compositions $\varphi\psi$ and $\psi\varphi$ are equal to f times the identity matrix. The pair (φ, ψ) is called the *reduced matrix factorization* corresponding to M, and we write $M = \mathrm{Coker}(\varphi, \psi)$.

Suppose now that N is another MCM R-module with no nonzero free summand, with corresponding reduced matrix factorization (φ', ψ'), and suppose $h : N \to \mathrm{syz}^1_R(M)$ is a homomorphism. Since the resolution of M is periodic of period 2 [16, Chapter 7], $\mathrm{syz}^1_R(M) = \mathrm{Coker}(\psi, \varphi)$. We can choose homomorphisms α and β to make the following diagram commute:

$$\begin{array}{ccccccc}
S^m & \xrightarrow{\varphi'} & S^m & \longrightarrow & N & \longrightarrow & 0 \\
\beta\downarrow & & \alpha\downarrow & & h\downarrow & & \\
S^n & \xrightarrow{\psi} & S^n & \longrightarrow & \mathrm{syz}^1_R(M) & \longrightarrow & 0.
\end{array}$$

Since M is its own second syzygy, we have an exact sequence

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow \operatorname{syz}^1_R(M) \longrightarrow 0. \tag{6}$$

Applying $\operatorname{Hom}_R(N, -)$ induces a surjection $\rho : \operatorname{Hom}_R(N, \operatorname{syz}^1_R(M)) \to \operatorname{Ext}^1_R(N, M)$ (recall that R is a hypersurface, hence Gorenstein, so $\operatorname{Ext}^1_R(N, R^n) = 0$). Now, the image of the map h under ρ can be represented by a short exact sequence $0 \to M \to L \to N \to 0$, which corresponds to a pullback of (6) by h.

Then L is a MCM R-module, and [16, 7.8] the reduced matrix factorization corresponding to L is $\left(\begin{bmatrix} \varphi & \beta \\ 0 & \varphi' \end{bmatrix}, \begin{bmatrix} \psi & -\alpha \\ 0 & \psi' \end{bmatrix}\right)$.

The (A_n) singularities, *n* even. Let $R = V[[y]]/(y^2 + \pi^{n+1})$, where $n \geq 2$ is an *even* integer. Assume that the residue field characteristic p is not equal to 2. Then R is a singularity of type (A_n). We will show that R has finite CM type. The polynomial $y^2 + \pi^{n+1}$ has no linear factors, since $n + 1$ is odd. Therefore, we have matrix factorizations of $y^2 + \pi^{n+1}$ of the form

$$\varphi_j = \begin{bmatrix} y & \pi^j \\ \pi^{n-j+1} & -y \end{bmatrix}, \qquad 0 \leq j \leq n+1, \tag{7}$$

and we will see that these are all the matrix factorizations up to equivalence. Set $M_j = \operatorname{Coker} \varphi_j$. Since elementary row and column operations transform φ_j into φ_{n-j+1}, $M_j \cong M_{n-j+1}$ for $0 \leq j \leq n/2$. Further, each M_j is indecomposable; a decomposition would lead to a linear factorization of f. Finally, note that $M_0 \cong R$, and each M_j is isomorphic to the ideal $(y, \pi^j)R$.

We now compute the AR sequence ending in M_j. Choose $j \geq 1$. Since M_j is isomorphic to its own first syzygy, we have an exact sequence

$$0 \longrightarrow M_j \longrightarrow R^2 \longrightarrow M_j \longrightarrow 0. \tag{8}$$

Consider the two endomorphisms of $M_j \cong (y, \pi^j)$ given by multiplication by $-y$ and multiplication by π^n. We claim that the pullback of (8) given by either of these endomorphisms is a split exact sequence. We need only show that each of these endomorphisms of M_j factors through the free module R^2. Define a map $M_j \to R^2$ by $x \mapsto \binom{-x}{0}$; then composition with the surjection $[\,y \ \pi^j\,] : R^2 \to M_j$ is equal to multiplication by $-y$. On the other hand, the map $M_j \to R^2$ taking $x \in M_j$ to $\binom{0}{\pi^{n-j}x}$ gives a factorization of the map given by multiplication by π^n through the free module R^2.

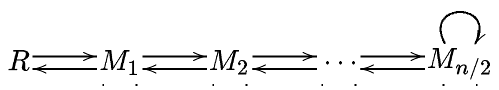
Let h be the endomorphism of M_j defined by multiplication by π^n/y, an element of the total quotient ring of R. Then yh is multiplication by π^n, and πh is multiplication by $-y$. By the previous paragraph, the images of yh and πh in $\operatorname{Ext}^1_R(M_j, M_j)$ are zero. This shows that the image of h is in the socle of $\operatorname{Ext}^1_R(M_j, M_j)$. If we show that the pullback of (8) by h is not a split sequence, then we will have shown that the image of h in $\operatorname{Ext}^1_R(M_j, M_j)$ generates the socle. We can take $\beta = -\alpha = \begin{bmatrix} 0 & \pi^{j-1} \\ -\pi^{n-j} & 0 \end{bmatrix}$ in (3) to represent h as a pair of maps between free modules. Thus pulling back by h gives a short exact sequence $0 \to M_j \to L \to M_j \to 0$, where

$$L = \operatorname{Coker} \begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix}.$$

It is a fairly straightforward matrix-equivalence computation to check that L is isomorphic to $M_{j-1} \oplus M_{j+1}$. To wit:

$$\begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix} \sim \begin{bmatrix} y & 0 & 0 & \pi^{j-1} \\ 0 & -y & -\pi^{n-j} & 0 \\ y\pi & -\pi^j & y & \pi^j \\ \pi^{n-j+2} & y\pi & \pi^{n-j+1} & -y \end{bmatrix} \sim$$
$$\begin{bmatrix} y & 0 & 0 & \pi^{j-1} \\ 0 & -y & -\pi^{n-j} & 0 \\ 0 & -\pi^j & y & 0 \\ \pi^{n-j+2} & 0 & 0 & -y \end{bmatrix} \sim \begin{bmatrix} y & \pi^{j-1} & 0 & 0 \\ \pi^{n-j+2} & -y & 0 & 0 \\ 0 & 0 & -y & -\pi^{n-j} \\ 0 & 0 & -\pi^j & y \end{bmatrix}$$

Since the result of the pullback by h is not split, the AR sequence ending in M_j is indeed $0 \to M_j \to M_{j-1} \oplus M_{j+1} \to M_j \to 0$. Hence we can draw a connected component of the AR quiver for R.



(A_n) for even n

By Theorem 2.9, this is the complete quiver. Thus R has finite CM type.

The ($\mathbf{A}_n$) singularities, *n* odd. Let $R = V[[y]]/(y^2 + \pi^{n+1})$, with n an *odd* positive integer. Assume that V has residue field characteristic greater than 2. Then V contains an element i such that $i^2 = -1$ (the residue field does, and use Hensel's Lemma to lift it back up to V). Now, R is no longer a domain, for we have $y^2 + \pi^{n+1} = (\pi^{(n+1)/2} + iy)(\pi^{(n+1)/2} - iy)$.

Set

$$\begin{aligned} N_+ &= R/(\pi^{(n+1)/2} + iy) \\ N_- &= R/(\pi^{(n+1)/2} - iy) \\ \varphi_j &= \begin{bmatrix} y & \pi^j \\ \pi^{n-j+1} & -y \end{bmatrix}, \quad 1 \le j \le n+1 \\ M_j &= \operatorname{Coker} \varphi_j. \end{aligned}$$

Then, as before, $M_j \cong M_{n-j+1}$ is an ideal for $j = 1, \dots, (n+1)/2$, and $M_0 \cong R$. Furthermore,

$$\varphi_{(n+1)/2} = \begin{bmatrix} y & \pi^{(n+1)/2} \\ \pi^{(n+1)/2} & -y \end{bmatrix} \sim$$
$$\begin{bmatrix} y - i\pi^{(n+1)/2} & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} & -y \end{bmatrix} \sim \begin{bmatrix} 0 & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} - iy & -y \end{bmatrix} \sim$$
$$\begin{bmatrix} 0 & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} - iy & -2y \end{bmatrix} \sim \begin{bmatrix} 0 & \pi^{(n+1)/2} - iy \\ \pi^{(n+1)/2} - iy & -y + i\pi^{(n+1)/2} \end{bmatrix} \sim$$
$$\begin{bmatrix} 0 & \pi^{(n+1)/2} - iy \\ \pi^{(n+1)/2} - iy & 0 \end{bmatrix}$$

so $M_{(n+1)/2} \cong N_+ \oplus N_-$.

Since they arise as matrix factorizations, N_+, N_-, and M_j are all MCM R-modules. The AR translations are given by $\tau(-) = \operatorname{syz}^1_R(-)$, so $\tau(M_j) \cong M_j$, $\tau(N_+) \cong N_-$, and $\tau(N_-) \cong N_+$.

As in the case where n is even, the AR sequence for M_j is $0 \to M_j \to L \to M_j \to 0$ where

$$L = \operatorname{Coker} \begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j+1} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix} \cong M_{j-1} \oplus M_{j+1}.$$

To compute the AR sequence ending in N_+, consider the endomorphism h of N_+ given by multiplication by $\pi^{(n-1)/2}$. We have $\pi h = \pi^{(n+1)/2}$ and $yh = y\pi^{(n-1)/2}$. Pulling back the short exact sequence $0 \to N_- \to R \to N_+ \to 0$ via $2iy$ gives a middle term with presentation matrix

$$\begin{bmatrix} \pi^{(n+1)/2} + iy & 2iy \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix} \sim \begin{bmatrix} \pi^{(n+1)/2} + iy & 0 \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix},$$

so multiplication by $2iy$ splits the resolution of N_+. It follows that y splits the sequence, since $2i$ is a unit, and so $y\pi^{(n-1)/2}$ does as well. We also have $yN_+ = y(\pi^{(n+1)/2} - iy)\pi^{(n+1)/2}(\pi^{(n+1)/2} - iy)$, so $yN_+ = \pi^{(n+1)/2}N_+$. This shows that the map $\pi h = \pi^{(n+1)/2}$ also splits the exact sequence $0 \to N_- \to R \to N_+ \to 0$. It remains only to show that pulling back via h does not split the sequence, and we will have that h generates the socle of $\operatorname{Ext}^1_R(N_+, N_-)$. The sequence obtained by pulling back via h is $0 \to N_- \to P \to N_+ \to 0$, where

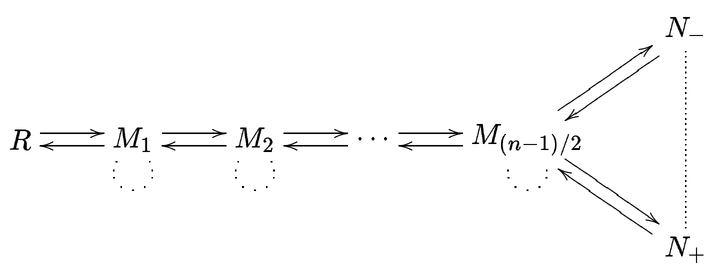
$$P = \operatorname{Coker} \begin{bmatrix} \pi^{(n+1)/2} + iy & \pi^{(n-1)/2} \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix} \cong$$

$$\operatorname{Coker} \begin{bmatrix} iy & \pi^{(n-1)/2} \\ -\pi^{(n-1)/2} + iy\pi & \pi^{(n+1)/2} - iy \end{bmatrix} \cong \operatorname{Coker} \begin{bmatrix} iy & \pi^{(n-1)/2-1} \\ -\pi^{(n-1)/2} & -iy \end{bmatrix} \cong$$

$$\cong M_{n/2-1}.$$

Since this is nonsplit, $0 \to N_- \to M_{(n-1)/2} \to N_+ \to 0$ is the AR sequence ending in N_+. Taking syzygies, we see that the AR sequence ending in N_- is $0 \to N_+ \to M_{(n-1)/2} \to N_- \to 0$.

Thus a connected component of the AR quiver for R is as follows.



(A_n) for odd n

By Theorem 2.9, this is the whole quiver, and so R has finite CM type. This completes the (A_n) singularities.

The ($\mathbf{A}'_n$) singularities. Let $R = V[[y]]/(\pi^2 + y^{n+1})$. Assume that the residue field characteristic p does not divide $n+1$. The matrix calculations of the previous section hold true if y and π are interchanged, so R has finite CM type.

The ($\mathbf{D}_n$) singularities, n even. Let $R = V[[y]]/(y^2\pi + \pi^{n-1})$, where $n \geq 4$ is an even integer. Assume that the residue field characteristic p is not equal to 2. As before, there is an element i in R such that $i^2 = -1$.

Since π is a zerodivisor in R, we cannot apply the methods of the previous section to draw a connected component of the AR quiver for R and conclude that it is the whole quiver using Theorem 2.9. Instead, to show that R has finite CM type we use a pair of criteria due to Drozd and Roĭter.

Theorem 3.3 ([14, Theorem 3.3])**.** *Let $(R, \mathfrak{m})$ be a one-dimensional reduced local ring whose integral closure $\widetilde{R}$ is finitely generated as an R-module. Let $\mu_R(M)$ denote the number of generators required for M as an R-module. Then R has finite CM type if and only if the following two conditions hold.*

(DR1) $\mu_R(\widetilde{R}) \leq 3$

(DR2) $\mu_R((\mathfrak{m}\widetilde{R} + R)/R) \leq 1.$

In our context, R is a complete one-dimensional reduced local ring, so $\widetilde{R}$ is always a finitely generated R-module [10, p. 264]. The condition (DR1) says simply that R has multiplicity at most 3, clearly true in this case. To verify (DR2), we must find generators for $\widetilde{R}$ as an R-module.

Note that since n is even, R has three minimal primes: $\mathfrak{p} = (\pi)R, \mathfrak{q} = (y + i\pi^{(n-2)/2})R$, and $\mathfrak{r} = (y - i\pi^{(n-2)/2})R$. The natural embedding of R into its total quotient ring $Q(R)$ factors through the product $T = R/\mathfrak{p} \times R/\mathfrak{q} \times R/\mathfrak{r}$. Since T is a module-finite extension of R which is contained in $Q(R)$, $\widetilde{R} = \widetilde{T}$. But

$$\widetilde{T} \cong (V/(\pi)[[y]] \times V[[y]]/(y + i\pi^{(n-1)/2}) \times V[[y]]/(y - i\pi^{(n-1)/2})$$

is a direct product of discrete valuation rings, so is integrally closed. So $\widetilde{R} = T$. As an R-submodule of $Q(R)$, T is minimally generated by $\{1, \frac{\pi^{n/2}}{y}, \frac{\pi^{n-2}}{y^2}\}$. It is now easily checked that $(\mathfrak{m}\widetilde{R} + R)/R$ requires exactly one generator, $\frac{\pi}{y}$. By Theorem 3.3, R has finite CM type.

The ($\mathbf{D}_n$) singularities, n odd. Let $R = V[[y]]/(y^2\pi + \pi^{n-1})$, where now $n \geq 4$ is an odd integer. We again use Theorem 3.3 to show that R has finite CM type.

Since n is odd, R has only two minimal primes, $\mathfrak{p} = (\pi)R$ and $\mathfrak{q} = (y^2 + \pi^{n-2})$. The integral closure of R is thus equal to the integral closure of $T = R/\mathfrak{p} \times R/\mathfrak{q}$. In this case, T is not integrally closed. The integral closure of $R/\mathfrak{q} = V[[y]]/(y^2 + \pi^{n-2})$ is easily see to be generated by $\frac{\pi^{(n+1)/2}}{y}$. Generators for $\widetilde{T}$ as an R-module are then $\{1, \frac{\pi^{(n+1)/2}}{y}, \frac{\pi^{n-2}}{y^2}\}$. As before, criterion (DR2) now easily verified, so R has finite CM type.

The ($\mathbf{D}'_n$) singularities, n odd. Let $R = V[[y]]/(y\pi^2 + y^{n-1})$, where $n \geq 4$ is an odd integer. Assume that the residue field characteristic p does not divide $n-2$. Set

$$\begin{aligned}
\alpha &= [y] \\
\beta &= [\pi^2 + y^{n-2}] \\
\varphi_j &= \begin{bmatrix} \pi & y^j \\ y^{n-j-2} & -\pi \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\psi_j &= \begin{bmatrix} y\pi & y^{j+1} \\ y^{n-j-1} & -y\pi \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\chi_j &= \begin{bmatrix} \pi & y^j \\ y^{n-j-1} & -y\pi \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\eta_j &= \begin{bmatrix} y\pi & y^j \\ y^{n-j-1} & -\pi \end{bmatrix}, \quad 0 \leq j \leq n-3.
\end{aligned} \tag{9}$$

It is easy to check that $(\alpha, \beta), (\beta, \alpha), (\varphi_j, \psi_j), (\psi_j, \varphi_j), (\chi_j, \eta_j), (\eta_j, \chi_j)$ are all matrix factorizations of $y\pi^2 + y^{n-1}$. Put

$$\begin{aligned}
A &= \operatorname{Coker} \alpha, & B &= \operatorname{Coker} \beta, \\
M_j &= \operatorname{Coker} \varphi_j, & N_j &= \operatorname{Coker} \psi_j, \\
X_j &= \operatorname{Coker} \chi_j, & Y_j &= \operatorname{Coker} \eta_j.
\end{aligned} \tag{10}$$

There is some collapsing here: $M_0 \cong B$, $N_0 \cong A \oplus R$, and $X_0 \cong Y_0 \cong R$. Also, $X_{(n-1)/2} \cong Y_{(n-1)/2}$, $M_j \cong M_{n-j-2}$, $N_j \cong N_{n-j-2}$, and $Y_j \cong X_{n-j-1}$ for $j = 0, \dots n-3$. Finally, before we compute the AR sequences, note that M_j is isomorphic to the ideal $(y\pi, y^{j+1})R$, and Y_j is isomorphic to $(\pi, y^j)R$.

Using the fact that $\tau(-) \cong \operatorname{syz}^1_R(-)$ by Lemma 1.2, we can see that our collection of modules is closed under AR translations. We now compute the AR sequences.

Note that B is isomorphic to the ideal $(y)R$. Consider the first part of a free resolution of B: $0 \to A \to R \to B \to 0$. The endomorphism of B given by multiplication by π^2 factors through the free module R via $x \mapsto -y^{n-2}x$. Similarly, the map on B given by multiplication by $y\pi$ factors through R via $x \mapsto \pi x$. Hence both of these endomorphisms of B split the resolution of B. We will show that the short exact sequence given by pulling back the map given by multiplication by π represents the socle element of $\operatorname{Ext}^1_R(B, A)$. The middle term of this sequence has presenting matrix $\left[\begin{smallmatrix} \beta & \pi \\ 0 & \alpha \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi^2+y^{n-2} & \pi \\ 0 & y \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} y^{n-2} & \pi \\ -y\pi & y \end{smallmatrix}\right]$. This is the presenting matrix for X_1, so the result of pulling back via π is nonsplit, and π maps to a nonzero socle element of $\operatorname{Ext}^1_R(B, A)$. Thus the AR sequence ending in B is $0 \to A \to X_1 \to B \to 0$. Taking syzygies gives the AR sequence ending in A: $0 \to B \to Y_1 \to A \to 0$.

On to the M_j's. Recall that $M_j \cong (y\pi, y^{j+1})R$. We have the exact sequence $0 \to N_j \to R^2 \to M_j \to 0$. The endomorphism of M_j given by multiplication by $-y\pi$ factors through the free module R^2 via $x \mapsto \left(\begin{smallmatrix} -x \\ 0 \end{smallmatrix}\right)$, while the map given by multiplication by π^2 factors through R^2 via $x \mapsto \left(\begin{smallmatrix} -y^{n-2}x \\ 0 \end{smallmatrix}\right)$. Thus multiplication by both $-y\pi$ and π^2 factor through R^2, and so pulling back by either of these splits a free resolution of M_j. If we show that pulling back by $-\pi$ does not split that resolution, we will have identified the element that generates the socle of $\operatorname{Ext}^1_R(M_j, N_j)$.

The map given by multiplication by $-\pi$ has a matrix factorization (γ, δ), where $\delta = -\gamma = \left[\begin{smallmatrix} 0 & y^j \\ -y^{n-j-2} & 0 \end{smallmatrix}\right]$ and so the middle term of the short exact sequence obtained by pulling back via $-\pi$ has presenting matrix

$$\begin{bmatrix} \varphi_j & \delta \\ 0 & \psi_j \end{bmatrix} \sim \begin{bmatrix} \pi & y^j & 0 & y^j \\ y^{n-j-2} & -\pi & -y^{n-j-2} & 0 \\ 0 & 0 & y\pi & y^{j+1} \\ 0 & 0 & y^{n-j-1} & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} \pi & y^j & 0 & y^j \\ y^{n-j-2} & -\pi & -y^{n-j-2} & 0 \\ -y\pi & -y^{j+1} & y\pi & 0 \\ 0 & 0 & y^{n-j-1} & -y\pi \end{bmatrix} \sim \begin{bmatrix} \pi & y^j & 0 & y^j \\ 0 & -\pi & -y^{n-j-2} & 0 \\ 0 & -y^{j+1} & y\pi & 0 \\ y^{n-j-1} & 0 & y^{n-j-1} & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} \pi & y^j & 0 & y^j \\ 0 & -\pi & -y^{n-j-2} & 0 \\ 0 & -y^{j+1} & y\pi & 0 \\ y^{n-j-1} & -y\pi & 0 & -y\pi \end{bmatrix} \sim \begin{bmatrix} \pi & 0 & 0 & y^j \\ 0 & -\pi & -y^{n-j-2} & 0 \\ 0 & -y^{j+1} & y\pi & 0 \\ y^{n-j-1} & 0 & 0 & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} \pi & y^{j+1} & 0 & 0 \\ y^{n-j-2} & -y\pi & 0 & 0 \\ 0 & 0 & y\pi & y^j \\ 0 & 0 & y^{n-j-1} & -\pi \end{bmatrix}$$

which is the presenting matrix for $X_{j+1} \oplus Y_j$. This shows that the image of $-\pi$ generates the socle of $\operatorname{Ext}^1_R(M_j, N_j)$, and the AR sequence ending in M_j is $0 \to N_j \to X_{j+1} \oplus Y_j \to M_j \to 0$. For N_j, we can just take syzygies in the AR sequence ending in M_j. This gives $0 \to M_j \to Y_{j+1} \oplus X_j \to N_j \to 0$ for the AR sequence ending in N_j.

Next we consider Y_j, which is isomorphic to the ideal $(\pi, y^j)R$. Pull back the resolution $0 \to X_j \to R^2 \to Y_j \to 0$ via the map given by multiplication by $y\pi$ on Y_j. Since $y\pi$ factors through R^2 via $x \mapsto \left(\begin{smallmatrix} xy \\ 0 \end{smallmatrix}\right)$, the result splits. Now pull back by the map given by multiplication by π^2. Again, $x \mapsto \left(\begin{smallmatrix} x\pi \\ 0 \end{smallmatrix}\right)$ is a map $Y_j \to R^2$ which factors π^2, so the result splits.

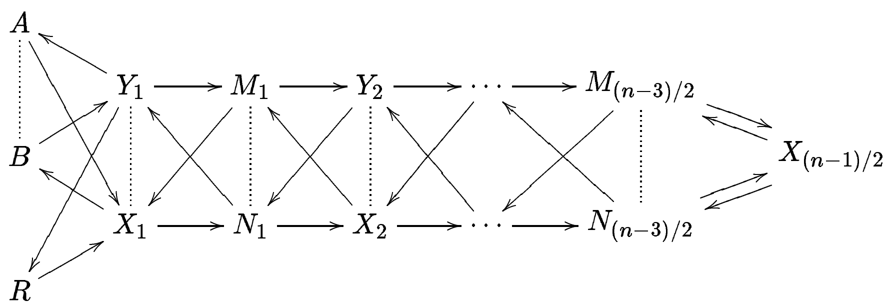
We now show that the result of pulling back by π is not split, so that multiplication by π on Y_j gives the socle element of $\operatorname{Ext}^1_R(Y_j, X_j)$, that is, the AR sequence ending in Y_j. We can write $\pi = \operatorname{Coker}(\gamma, \delta)$, where $\gamma = \left(\begin{smallmatrix} 0 & y^j \\ -y^{n-j-2} & \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} 0 & -y^{j-1} \\ y^{n-j-1} & 0 \end{smallmatrix}\right)$. A presenting matrix for the middle term of the extension obtained by pulling back by π is thus

$$\begin{bmatrix} \pi & y^j & 0 & -y^{j-1} \\ y^{n-j-1} & -y\pi & y^{n-j-1} & 0 \\ 0 & 0 & y\pi & y^j \\ 0 & 0 & y^{n-j-1} & \pi \end{bmatrix} \sim \begin{bmatrix} \pi & y^j & 0 & -y^{j-1} \\ y^{n-j-1} & -y\pi & y^{n-j-1} & 0 \\ y\pi & -y^{j+1} & y\pi & 0 \\ -y^{n-j-1} & y\pi & 0 & -\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} \pi & 0 & 0 & -y^{j-1} \\ 0 & -y\pi & y^{n-j-1} & 0 \\ 0 & y^{j+1} & y\pi & 0 \\ -y^{n-j-1} & 0 & 0 & -\pi \end{bmatrix} \sim \begin{bmatrix} \pi & -y^{j-1} & 0 & 0 \\ -y^{n-j-1} & -\pi & 0 & 0 \\ 0 & 0 & -y\pi & y^{n-j-1} \\ 0 & 0 & y^{j+1} & y\pi \end{bmatrix}$$

This is the presentation matrix for $M_{j-1} \oplus N_j$, so the AR sequence ending in Y_j is $0 \to X_j \to M_{j-1} \oplus N_j \to Y_j \to 0$. As before, we take syzygies to see that the AR sequence ending in X_j is $0 \to Y_j \to N_{j-1} \oplus M_j \to X_j \to 0$.

This allows us to draw a connected component of the AR quiver for R.



(D'_n) for odd n

The ($\mathbf{D}'_n$) singularities, n even. Let $R = V[[y]]/(y\pi^2 + y^{n-1})$, where $n \geq 4$ is an even integer. Assume the residue field characteristic p is not 2. Then, as in the A_n singularities with n even, V contains an element i whose square is -1. Define A, B, M_j, N_j, X_j, and Y_j as in (9) and (10). Also let

$$
\begin{aligned}
C_+ &= \mathrm{Coker}(y(\pi + iy^{(n-2)/2})) \\
C_- &= \mathrm{Coker}(y(\pi - iy^{(n-2)/2})) \\
D_= &= \mathrm{Coker}(\pi - iy^{(n-2)/2}) \\
D_+ &= \mathrm{Coker}(\pi + iy^{(n-2)/2})
\end{aligned}
\tag{11}
$$

Then, as in the case of n odd, $M_0 \cong B \oplus R$, $N_0 \cong A$, and $X_0 \cong Y_0 \cong R$. Also, $X_{(n-1)/2} \cong Y_{(n-1)/2}$, $M_j \cong M_{n-j-2}$, $N_j \cong N_{n-j-2}$, $X_j \cong Y_{n-j-1}$, and $Y_j \cong X_{n-j-1}$. Furthermore, M_j is isomorphic to the ideal $(y\pi, y^{j+1})R$, and Y_j is isomorphic to $(\pi, y^j)R$. In this case, however, $M_{(n-2)/2} \cong D_+ \oplus D_-$ and $N_{(n-2)/2} \cong C_+ \oplus C_-$.

We already know that we have AR sequences

$$
\begin{aligned}
0 \to A \to &X_1 \to B \to 0 \\
0 \to B \to &Y_1 \to A \to 0
\end{aligned}
\tag{12}
$$

and, for $j \neq (n-2)/2$,

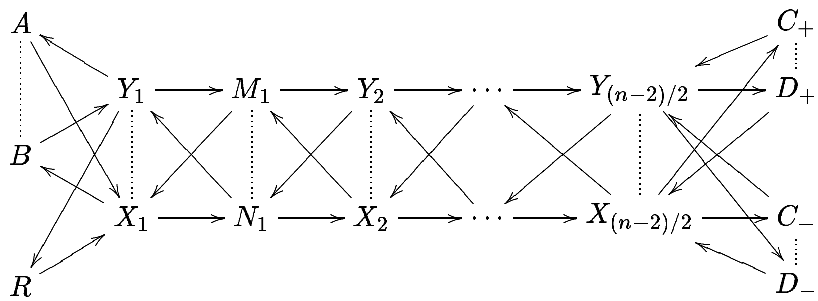
$$
\begin{aligned}
0 &\to N_j \to X_{j+1} \oplus Y_j \to M_j \to 0 \\
0 &\to M_j \to Y_{j+1} \oplus X_j \to N_j \to 0 \\
0 &\to X_j \to M_{j-1} \oplus N_j \to Y_j \to 0 \\
0 &\to Y_j \to N_{j-1} \oplus M_j \to X_j \to 0.
\end{aligned}
\tag{13}
$$

All that remains is to compute the AR sequences ending in $C_\pm$ and $D_\pm$. Clearly the AR translation of $C_\pm$ is $D_\pm$, and vice versa. From the decompositions $M_{(n-2)/2} \cong D_+ \oplus D_-$ and

$N_{(n-2)/2} \cong C_+ \oplus C_-$ we get AR sequences

$$\begin{aligned} 0 \to D_+ \to & X_{(n-2)/2} \to C_+ \to 0 \\ 0 \to D_- \to & X_{(n-2)/2} \to C_- \to 0 \\ 0 \to C_+ \to & Y_{(n-2)/2} \to D_+ \to 0 \\ 0 \to C_- \to & Y_{(n-2)/2} \to D_- \to 0 \end{aligned}$$

Note that $Y_{(n-2)/2} \cong X_{n/2}$, so we get the following connected component of the AR quiver.



(D'_n) for even n

The ($\mathbf{E}_6$) singularity. Let $R = V[[y]]/(y^3 + \pi^4)$, assume $p \neq 3$, and let

$$\begin{aligned} \varphi_1 &= \begin{bmatrix} y & \pi \\ \pi^3 & -y^2 \end{bmatrix} & \psi_1 &= \begin{bmatrix} y^2 & \pi \\ \pi^3 & -y \end{bmatrix} \\ \varphi_2 &= \begin{bmatrix} y & \pi^2 \\ \pi^2 & -y^2 \end{bmatrix} & \psi_2 &= \begin{bmatrix} y^2 & \pi^2 \\ \pi^2 & -y \end{bmatrix} \\ \alpha &= \begin{bmatrix} \pi^3 & y^2 & y\pi^2 \\ y\pi & -\pi^2 & y^2 \\ y^2 & -y\pi & -\pi^3 \end{bmatrix} & \beta &= \begin{bmatrix} \pi & 0 & y \\ y & -\pi^2 & 0 \\ 0 & y & -\pi \end{bmatrix} \\ \chi &= \begin{bmatrix} \varphi_2 & \begin{bmatrix} 0 & \pi \\ -y\pi & 0 \end{bmatrix} \\ 0 & \psi_2 \end{bmatrix} & \eta &= \begin{bmatrix} \psi_2 & \begin{bmatrix} 0 & y\pi \\ -\pi & 0 \end{bmatrix} \\ 0 & \varphi_2 \end{bmatrix} \end{aligned}$$

Then each pair (φ_i, ψ_i), (α, β), (χ, η) is a matrix factorization of $y^3 + \pi^4$. Let $M_i = \operatorname{Coker} \varphi_i$, $N_i = \operatorname{Coker} \psi_i$, $A = \operatorname{Coker} \alpha$, $B = \operatorname{Coker} \beta$, $X = \operatorname{Coker} \chi$, $Y = \operatorname{Coker} \eta$.

We can identify these modules more clearly. It is easy to check that $M_1 \cong (y^2, \pi)R$, $N_1 \cong (y, \pi)R$, $N_2 \cong M_2 \cong (y^2, \pi^2)R$, $B \cong (y^2, y\pi, \pi^2)R$, and A has rank 2. Also, $X \cong Y$. Using the fact that $\tau(-) = \operatorname{syz}^1_R(-)$ we can see that our collection of modules is closed under AR translations. We now compute the AR sequences.

Begin with $N_1 \cong (y, \pi)R$. A presentation of N_1 is given by $0 \to M_1 \to R^2 \to N_1 \to 0$. When we pull back along the endomorphism of N_1 given by multiplication by $-\pi^3$, we see that the map $N_j \to R^2$ given by $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi^2 x \end{smallmatrix}\right)$ factors $-\pi^3$ through a free module. Similarly, the map given by multiplication by y^2 on N_j factors through R^2 via $x \mapsto \left(\begin{smallmatrix} xy \\ 0 \end{smallmatrix}\right)$.

We will show that the endomorphism h of N_1 given by multiplication by $-\pi^3/y$ gives the socle element of $\operatorname{Ext}^1_R(N_1, M_1)$, that is, the AR sequence ending in N_1. Note that $yh = -\pi^3$ and $\pi h = y^2$, so we need only show that pulling back by h does not split the short exact sequence

$0 \to M_1 \to R^2 \to N_1 \to 0$. We can write $h = \mathrm{Coker}(\gamma, \delta)$, where $\gamma = \left[\begin{smallmatrix} 0 & 1 \\ -\pi^2 & 0 \end{smallmatrix}\right]$ and $\delta = \left[\begin{smallmatrix} 0 & -1 \\ y\pi^2 & 0 \end{smallmatrix}\right]$. A presenting matrix for the middle term of the AR sequence ending in N_1 is

$$\begin{bmatrix} \varphi_1 & \begin{bmatrix} 0 & -1 \\ y\pi^2 & 0 \end{bmatrix} \\ 0 & \psi_1 \end{bmatrix} \sim \begin{bmatrix} y & \pi & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ 0 & 0 & y^2 & \pi \\ 0 & 0 & \pi^3 & -y \end{bmatrix} \sim$$
$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ -y\pi & -\pi^2 & y^2 & \pi \\ y^2 & y\pi & \pi^3 & -y \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ -y\pi & -\pi^2 & y^2 & 0 \\ y^2 & y\pi & \pi^3 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^3 & y^2 & y\pi^2 \\ 0 & y\pi & -\pi^2 & y^2 \\ 0 & y^2 & -y\pi & -\pi^3 \end{bmatrix}$$

Since this is the presenting matrix for A, the AR sequence ending in N_1 is $0 \to M_1 \to A \to N_1 \to 0$. As always, we take syzygies to see that the AR sequence ending in M_1 is $0 \to N_1 \to B \oplus R \to M_1 \to 0$.

Now we compute the AR sequence ending in $M_2 \cong (y^2, \pi^2)R$. The resolution of M_2 starts out $0 \to M_2 \to R^2 \to M_2 \to 0$. Pull back along the endomorphism of M_2 given by multiplication by y^2. This map factors through R^2, using $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for the splitting map $M_2 \to R^2$. Similarly, the map $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi x \end{smallmatrix}\right)$ factors the map given by multiplication by $-\pi^3$ through R^2.

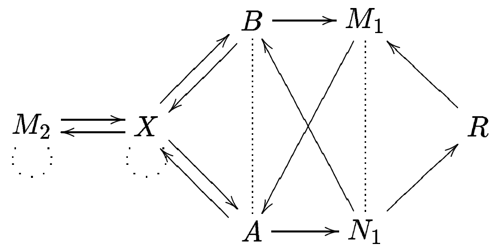
These two maps are πg and yg, respectively, where g is the endomorphism of M_2 given by multiplication by $-\pi^3/y$. We will show that g gives the AR sequence ending in M_2. The map g has associated matrix factorization $(\gamma, \delta) = \left(\left[\begin{smallmatrix} 0 & \pi \\ -y\pi & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ \pi & 0 \end{smallmatrix}\right]\right)$, so the middle term of the extension obtained by pulling back via g has presenting matrix $\eta = \left[\begin{smallmatrix} \psi_2 & \delta \\ 0 & \varphi_2 \end{smallmatrix}\right]$. Since this extension does not split, g gives the socle element of $\mathrm{Ext}^1_R(M_2, M_2)$, and the AR sequence ending in M_2 is $0 \to M_2 \to X \to M_2 \to 0$. (Recall that $X \cong Y$.)

To finish the AR quiver, we use a process of elimination and count ranks. Consider the AR sequence ending in A. The third term, A, has rank two and its first syzygy, B, has rank one. The middle term, then, must have rank three. It has a summand isomorphic to M_1, and the complement U must have rank two. Considering the AR sequence for B, we see that the first syzygy of U must have rank two as well. The AR sequences for N_1 and M_2 don't involve A, so U has no summand isomorphic to either of those. The only option left is $U \cong X$. This gives the AR sequence ending in A: $0 \to B \to X \oplus M_1 \to A \to 0$. The AR sequence ending in B is, taking first syzygies, $0 \to A \to X \oplus N_1 \to B \to 0$.

Finally, the middle term of the AR sequence ending in X must have rank four, and involves summands isomorphic to A, B, and M_2. Hence the AR sequence is

$$0 \to X \to A \oplus B \oplus M_2 \to X \to 0.$$

The complete AR quiver is then as follows. We again use Theorem 2.9 to conclude that it is indeed the whole quiver.



(E_6)

The $(\mathbf{E}_6')$ singularity. Let $R = V[[y]]/(\pi^3 + y^4)$, and assume that the residue field characteristic p is greater than 2. This case is exactly symmetric to the (E_6) case just completed. So R has finite CM type.

The $(\mathbf{E}_7)$ singularity. Let $R = V[[y]]/(y^3 + y\pi^3)$. Assume that the residue field characteristic p is not equal to 2. Define seven pairs of matrices over R:

$$
\begin{aligned}
\alpha &= [y] & \beta &= [y^2+\pi^3] \\
\gamma &= \begin{bmatrix} y^2 & y\pi \\ y\pi^2 & -y^2 \end{bmatrix} & \delta &= \begin{bmatrix} y & \pi \\ \pi^2 & -y \end{bmatrix} \\
\varphi_1 &= \begin{bmatrix} y & \pi \\ y\pi^2 & -y^2 \end{bmatrix} & \psi_1 &= \begin{bmatrix} y^2 & \pi \\ y\pi^2 & -y \end{bmatrix} \\
\varphi_2 &= \begin{bmatrix} y & \pi^2 \\ y\pi & -y^2 \end{bmatrix} & \psi_2 &= \begin{bmatrix} y^2 & \pi^2 \\ y\pi & -y \end{bmatrix} \\
\chi_1 &= \begin{bmatrix} y\pi^2 & -y^2 & -y^2\pi \\ y\pi & \pi^2 & -y^2 \\ y^2 & y\pi & y\pi^2 \end{bmatrix} & \eta_1 &= \begin{bmatrix} \pi & 0 & y \\ -y & y\pi & 0 \\ 0 & -y & \pi \end{bmatrix} \\
\chi_2 &= \begin{bmatrix} y^2 & -\pi^2 & -y\pi \\ y\pi & y & -\pi^2 \\ y\pi^2 & y\pi & y^2 \end{bmatrix} & \eta_2 &= \begin{bmatrix} y & 0 & \pi \\ -y\pi & y^2 & 0 \\ 0 & -y\pi & y \end{bmatrix} \\
\chi_3 &= \begin{bmatrix} y^2 & y\pi & \pi & 0 \\ y\pi^2 & -y^2 & 0 & \pi \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^2 & -y \end{bmatrix} & \eta_3 &= \begin{bmatrix} y & \pi & -\pi & 0 \\ \pi^2 & -y & 0 & -\pi \\ 0 & 0 & y^2 & y\pi \\ 0 & 0 & y\pi^2 & -y^2 \end{bmatrix}
\end{aligned}
$$

As usual, we put

$$
\begin{aligned}
A &= \operatorname{Coker}\alpha, & B &= \operatorname{Coker}\beta \\
C &= \operatorname{Coker}\gamma, & D &= \operatorname{Coker}\delta \\
M_j &= \operatorname{Coker}\varphi_j, & N_j &= \operatorname{Coker}\psi_j \\
X_j &= \operatorname{Coker}\chi_j, & Y_j &= \operatorname{Coker}\eta_j.
\end{aligned}
\tag{14}
$$

It is an easy exercise to calculate the multiplicities of these modules, using, for instance, [10, 14.7]. This will be useful later. We obtain

$$
\begin{array}{ll}
e(A)=1 & e(B)=2 \\
e(C)=4 & e(D)=2 \\
\multicolumn{2}{l}{e(M_1)=e(N_1)=e(M_2)=e(N_2)=3} \\
e(X_1)=6 & e(Y_1)=3 \\
e(X_2)=5 & e(Y_2)=4 \\
e(X_3)=6 & e(Y_3)=6.
\end{array} \tag{15}
$$

The ring itself clearly has multiplicity 3. First we compute the AR sequence ending in A. The beginning of a resolution of A is $0 \to B \to R \to A \to 0$. Since $A \cong R/(y)$, pulling back along the map given by multiplication by y certainly splits this exact sequence. Since y^2 kills A, we see that $\pi^3 A = (y^2+\pi^3)A$. Therefore the map on A given by multiplication by π^3 factors through R by sending $x \in A$ to $x \in R$, which then goes to $(y^2+\pi^3)x = \pi^3 x$. We will show that the socle of $\operatorname{Ext}^1_R(A,B)$ is generated by the image of $\pi^2 \in \operatorname{Hom}_R(A,A)$. A presentation matrix for the middle term of the exact sequence obtained from π^2 is $\left[\begin{smallmatrix} y^2+\pi^3 & \pi^2 \\ 0 & y \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} y^2 & \pi^2 \\ y\pi & -y \end{smallmatrix}\right]$, which is the presenting matrix for N_2. This exact sequence does not split, so the AR sequence ending in A is thus $0 \to B \to N_2 \to A \to 0$. Taking syzygies gives the AR sequence ending in B: $0 \to A \to M_2 \to B \to 0$.

Next consider the AR sequence ending in $D \cong (y^2, y\pi)R$. We will show that the image of π generates the socle of $\operatorname{Ext}^1_R(D,C)$. Multiplication by π^2 on D factors through R^2 via the map $D \to R^2$ given by $x \mapsto \left(\begin{smallmatrix} 0 \\ -x/\pi^2 \end{smallmatrix}\right)$. Similarly, multiplication by y on D admits a factorization $x \mapsto \left(\begin{smallmatrix} -x/\pi^3 \\ 0 \end{smallmatrix}\right)$ through the free module R^2. Thus both these maps give the zero element of $\operatorname{Ext}^1_R(D,C)$.

Now, pulling back by π gives a nonsplit extension with middle term

$$
\chi_3 = \begin{bmatrix} y^2 & y\pi & \pi & 0 \\ y\pi^2 & -y^2 & 0 & \pi \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^2 & -y \end{bmatrix},
$$

so the AR sequence ending in D is $0 \to C \to X_3 \to D \to 0$. Taking syzygies shows that the AR sequence ending in C is $0 \to D \to Y_3 \to C \to 0$.

Next we compute the AR sequence ending in $M_1 \cong (y^2,\pi)R$. Consider the endomorphism h of M_1 given by multiplication by y^2/π. Then yh is multiplication by $-y\pi^2$, and πh is multiplication by y^2, both of which split a free resolution of M_1, as we shall show.

The map on M_1 given by $-y\pi^2$ factors through R^2 via $x \mapsto \left(\begin{smallmatrix} 0 \\ -y\pi x \end{smallmatrix}\right)$, so gives the zero element of $\operatorname{Ext}^1_R(M_1,N_1)$. The map given by multiplication by y^2 admits a factorization $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ through R^2.

So both of the maps yh and πh result in split exact sequences. All that remains is to show that h does not split a resolution of M_1. The map h is represented by the pair of matrices $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -y\pi^2 & 0 \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 0 & -y \\ y\pi & 0 \end{smallmatrix}\right]$. A presenting matrix for the middle term of the extension obtained by pulling back by h is then

$$\begin{bmatrix} y^2 & \pi & 0 & -y \\ y\pi^2 & -y & y\pi & 0 \\ 0 & 0 & y & \pi \\ 0 & 0 & y\pi^2 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \pi & 0 & y \\ 0 & -y & y\pi & 0 \\ 0 & 0 & -y & \pi \end{bmatrix}.$$

This is the presenting matrix for $R \oplus Y_1$, so the extension obtained by pulling back by h is not split. Hence the middle term of the AR sequence ending in M_1 is $R \oplus Y_1$, and the AR sequence is $0 \to N_1 \to R \oplus Y_1 \to M_1 \to 0$. Taking syzygies give the AR sequence ending in N_1, $0 \to M_1 \to X_1 \to N_1 \to 0$.

Continue with $M_2 \cong (y^2, \pi^2)R$. Define an endomorphism h of M by multiplication by y^2/π. Then yh is multiplication by $-y\pi^2$, and πh is multiplication by y^2. The map given by multiplication by $-y\pi^2$ admits the factorization $x \mapsto \left(\begin{smallmatrix} 0 \\ -yx \end{smallmatrix}\right)$ through the free module R^2. Similarly, the map $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ from M_2 to R^2 factors multiplication by y^2. To show that h goes to the socle element of $\operatorname{Ext}^1_R(M_2, N_2)$, then, we need only show that h does not split a free resolution of M_2. A matrix factorization representing h is $\left(\left[\begin{smallmatrix} 0 & \pi \\ -y^2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ y & 0 \end{smallmatrix}\right]\right)$, and so the middle term of the sequence obtained from h is presented by the matrix

$$\begin{bmatrix} y^2 & \pi^2 & 0 & -y\pi \\ y\pi & -y & y & 0 \\ 0 & 0 & y & \pi^2 \\ 0 & 0 & y\pi & -y^2 \end{bmatrix} \sim \begin{bmatrix} y & 0 & 0 & 0 \\ 0 & y^2 & -\pi^2 & -y\pi \\ 0 & y\pi & y & -\pi^2 \\ 0 & y\pi^2 & y\pi & y^2 \end{bmatrix}.$$

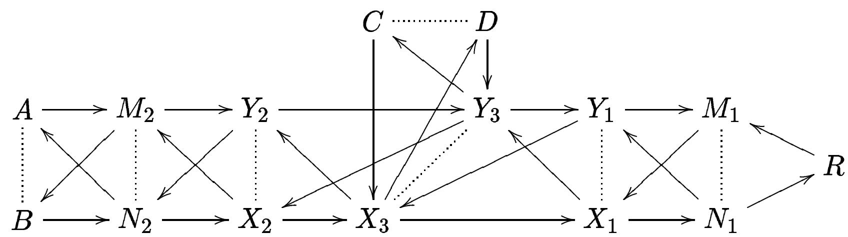
This gives the AR sequence ending in M_2, $0 \to N_2 \to A \oplus X_2 \to M_2 \to 0$, and the AR sequence ending in N_2 by taking syzygies: $0 \to M_2 \to B \oplus Y_2 \to N_2 \to 0$.

In order to compute the rest of the AR sequences, we refer to the multiplicities calculated earlier. Consider the AR sequence ending in Y_3. Since Y_3 and its first syzygy, X_3, both have multiplicity 6, the middle term of the AR sequence must have multiplicity 12. We know that is isomorphic to $D \oplus U$ for some U, which must have multiplicity 10 and not involve any of the other modules we have considered so far. Furthermore, its first syzygy must have also have multiplicity 10. It is an easy process of elimination to see that the only possibility is $U \cong X_1 \oplus Y_2$. Hence the AR sequence ending in Y_3 is $0 \to X_3 \to D \oplus X_1 \oplus Y_2 \to Y_3 \to 0$. Taking syzygies gives the AR sequence ending in X_3: $0 \to Y_3 \to C \oplus Y_1 \oplus X_2 \to X_3 \to 0$.

Next consider the AR sequence ending in X_1. The middle term must have multiplicity 9. We already have arrows to X_1 from X_3 and M_1, and $M_1 \oplus X_3$ has multiplicity 9. This gives the AR sequence $0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0$. Similar reasoning gives the AR sequence ending in Y_1: $0 \to X_1 \to N_1 \oplus Y_3 \to Y_1 \to 0$.

Finally, the middle term of the AR sequence ending in X_2 has multiplicity 9, and has direct summands isomorphic to N_2 and Y_3, so is isomorphic to $N_2 \oplus Y_3$. Taking syzygies for the AR sequence ending in Y_2, we get $0 \to Y_2 \to N_2 \oplus Y_3 \to X_2 \to 0$ and $0 \to X_2 \to M_2 \oplus X_3 \to Y_2 \to 0$.

This completes a connected component of the AR quiver for R, and so we have the whole quiver by Theorem 2.9.



(E_7)

The (E_7') singularity. Set $R = V[[y]]/(\pi^3 + y^3\pi)$. The matrix calculations of the previous section remain valid with y and π interchanged, but since π is a zerodivisor in R, we cannot apply Theorem 2.9 to conclude that R has finite CM type. Instead, we use Theorem 3.3 as in the case of the (D_n) singularities.

The minimal primes of R are $\mathfrak{p} = (\pi)R$ and $\mathfrak{q} = (\pi^2 + y^3)R$. Put $T = R/\mathfrak{p} \times R/\mathfrak{q}$. Then the embedding $R \hookrightarrow T$ induces an embedding $\widetilde{R} \hookrightarrow \widetilde{T} = \widetilde{R/\mathfrak{p}} \times \widetilde{R/\mathfrak{q}}$. Since $R/\mathfrak{p} \cong (V/(\pi))[[y]]$, $R/\mathfrak{p}$ is integrally closed. The integral closure of $R/\mathfrak{q} \cong V[[y]]/(\pi^2 + y^3)$ is generated over $R/\mathfrak{q}$ by $\frac{\pi}{y}$. Generators for $\widetilde{T}$ as an R-submodule of $Q(R)$ are thus $\{1, \frac{\pi}{y}, \frac{\pi^2}{y^3}\}$. Criterion (DR2) is easily checked, and so R has finite CM type.

The (E_8) singularity. Let $R = V[[y]]/(y^3 + \pi^5)$ with $p \neq 3$, a simple singularity of type (E_8). Define matrices over R:

$$\varphi_1 = \begin{bmatrix} y & \pi \\ \pi^4 & -y^2 \end{bmatrix} \qquad \psi_1 = \begin{bmatrix} y^2 & \pi \\ \pi^4 & -y \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} y & \pi^2 \\ \pi^3 & -y^2 \end{bmatrix} \qquad \psi_2 = \begin{bmatrix} y^2 & \pi^2 \\ \pi^3 & -y \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} \pi & -y & 0 \\ 0 & \pi & -y \\ y & 0 & \pi^3 \end{bmatrix} \qquad \beta_1 = \begin{bmatrix} \pi^4 & y\pi^3 & y^2 \\ -y^2 & \pi^4 & y\pi \\ -y\pi & -y^2 & \pi^2 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} \pi & -y & 0 \\ 0 & \pi^2 & -y \\ y & 0 & \pi^2 \end{bmatrix} \qquad \beta_2 = \begin{bmatrix} \pi^4 & y\pi^2 & y^2 \\ -y^2 & \pi^3 & y\pi \\ -y\pi^2 & -y^2 & \pi^3 \end{bmatrix}$$

$$\gamma_1 = \begin{bmatrix} \pi & y & 0 & \pi^3 \\ y & 0 & -\pi^3 & 0 \\ -\pi^3 & 0 & -y^2 & 0 \\ 0 & -\pi^2 & -y\pi & -y^2 \end{bmatrix} \qquad \delta_1 = \begin{bmatrix} 0 & y^2 & -\pi^3 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 \\ 0 & -\pi^2 & -y & 0 \\ \pi^2 & 0 & \pi & -y \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} \varphi_2 & \begin{pmatrix} 0 & \pi \\ -y\pi^2 & 0 \end{pmatrix} \\ 0 & \psi_2 \end{bmatrix} \qquad \delta_2 = \begin{bmatrix} \psi_2 & \begin{pmatrix} 0 & y\pi \\ -\pi^2 & 0 \end{pmatrix} \\ 0 & \varphi_2 \end{bmatrix}$$

$$\chi_1 = \begin{bmatrix} \beta_2 & \begin{pmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{pmatrix} \\ 0 & \alpha_2 \end{bmatrix} \qquad \eta_1 = \begin{bmatrix} \alpha_2 & \begin{pmatrix} 0 & 0 & -y \\ y\pi & 0 & 0 \\ 0 & y\pi & 0 \end{pmatrix} \\ 0 & \beta_2 \end{bmatrix}$$

$$\chi_2 = \begin{bmatrix} \pi^4 & y^2 & 0 & -y\pi^2 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 & 0 \\ 0 & -\pi^2 & -y & 0 & \pi^3 \\ -y\pi^2 & \pi^3 & 0 & y^2 & 0 \\ -\pi^3 & 0 & -\pi^2 & y\pi & -y^2 \end{bmatrix} \qquad \eta_2 = \begin{bmatrix} \pi & -y & 0 & 0 & 0 \\ y & 0 & 0 & \pi^2 & 0 \\ -\pi^2 & 0 & -y^2 & 0 & -\pi^3 \\ 0 & -\pi^2 & 0 & y & 0 \\ 0 & 0 & \pi^2 & \pi & -y \end{bmatrix}$$

As always, we associate modules to these matrices by $M_j = \operatorname{Coker}\varphi_j$, $N_j = \operatorname{Coker}\psi_j$, $A_j = \operatorname{Coker}\alpha_j$, $B_j = \operatorname{Coker}\beta_j$, $C_j = \operatorname{Coker}\gamma_j$, $D_j = \operatorname{Coker}\delta_j$, $X_j = \operatorname{Coker}\chi_j$, $Y_j = \operatorname{Coker}\eta_j$ for $j = 1, 2$.

Some of these are ideals: $M_1 \cong (y^2, \pi)R$, $M_2 \cong (y^2, \pi^2)R$, $N_1 \cong (y, \pi)R$, $N_2 \cong (y, \pi^2)R$, $A_1 \cong (y^2, y\pi^3, \pi^4)R$, and $A_2 \cong (y^2, y\pi^2, \pi^4)R$. The C_i and D_i all have rank 2, as does Y_2. The remaining modules, X_1, X_2, and Y_1, have rank 3.

Let's compute the AR sequences ending in these modules, starting with $M_1 = (y^2, \pi)R$. Starting from a free presentation of M_1, $0 \to N_1 \to R^2 \to M_1 \to 0$, pull back along the endomorphisms of M_1 given by multiplication by $-\pi^4$ and by y^2 to get split extensions. Multiplication by y^2 factors through R^2 via $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$, while multiplication by $-\pi^4$ factors through R^2 with $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi^3 x \end{smallmatrix}\right)$.

The maps y^2 and $-\pi^4$ are πh and yh, respectively, where h is the endomorphism of M_1 defined by multiplication by y^2/π. We can show that h is in the socle of $\operatorname{Ext}^1_R(M_1, N_1)$, and hence gives the AR sequence ending in M_1. Write $h = \operatorname{Coker}(\alpha, \beta)$, where $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -y\pi^3 & 0 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 0 & -y \\ \pi^3 & 0 \end{smallmatrix}\right]$. To identify the middle term of the extension given by h, consider

$$\begin{bmatrix} y^2 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^4 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ y\pi & 0 & y & \pi \\ \pi^5 & 0 & \pi^4 & -y^2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ y\pi & 0 & y & \pi \\ 0 & y\pi & 0 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & 0 & -y \\ 0 & -y & \pi^3 & 0 \\ 0 & 0 & y & \pi \\ 0 & y\pi & 0 & -y^2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi & 0 & y \\ 0 & -y & \pi^3 & 0 \\ 0 & y & \pi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & -y & 0 \\ 0 & 0 & \pi & -y \\ 0 & y & 0 & \pi^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the defining matrix for $A_1 \oplus R$, so we get the AR sequence ending in M_1: $0 \to N_1 \to A_1 \oplus R \to M_1 \to 0$. Taking syzygies implies that the AR sequence ending in N_1 is $0 \to M_1 \to B_1 \to N_1 \to 0$.

Now consider the first part of a resolution of $M_2 \cong (y^2, \pi^2)R$: $0 \to N_2 \to R^2 \to M_2 \to 0$. Pull back by the map given by multiplication by π^4 on M_2. This admits a factorization $x \mapsto \left(\begin{smallmatrix} 0 \\ \pi^2 x \end{smallmatrix}\right)$ through R^2, so results in a split sequence. Similarly, the map on M_1 given by multiplication by $-y^2$ factors through R^2 with $x \mapsto \left(\begin{smallmatrix} -x \\ 0 \end{smallmatrix}\right)$.

Let h be the endomorphism of M_2 given by multiplication by π^4/y. Then yh is multiplication by π^4 and πh is multiplication by $-y^2$ on M_2. Both of these endomorphisms give split exact sequences, so if h does not give a split sequence, then h gives the socle element of $\operatorname{Ext}^1_R(M_2, N_2)$. To identify the middle term of the sequence obtained by pulling back by h, note that a matrix factorization for h is $\left(\left[\begin{smallmatrix} 0 & \pi \\ -y\pi^3 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ -\pi^2 & 0 \end{smallmatrix}\right]\right)$, so the middle term of the sequence is given by the matrix δ_2, which is a presentation matrix for D_2, and so the AR sequence is $0 \to N_2 \to D_2 \to M_2 \to 0$. Again, syzygies give us that the AR sequence ending in N_2 is $0 \to M_2 \to C_2 \to N_2 \to 0$.

Moving along, we consider a free resolution of A_1: $0 \to B_1 \to R^3 \to A_1 \to 0$. Let h be the endomorphism of A_1 given by multiplication by $y^2\pi$; then yh is multiplication by $-\pi^4$ and πh is multiplication by y^2. Both of these maps factor through the free module R^3, via

$$x \mapsto \begin{pmatrix} 0 \\ 0 \\ -x \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix},$$

respectively. Thus both πh and yh induce split sequences. We can write

$$h = \operatorname{Coker}\left(\begin{bmatrix} 0 & 0 & \pi^3 \\ -\pi & 0 & 0 \\ 0 & \pi^3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -y\pi^2 \\ y & 0 & 0 \\ 0 & y & 0 \end{bmatrix} \right)$$

A matrix computation shows that

$$\begin{bmatrix} \pi^4 & y\pi^3 & y^2 & 0 & 0 & -y\pi^2 \\ -y^2 & \pi^4 & y\pi & y & 0 & 0 \\ -y\pi & -y^2 & \pi^2 & 0 & y & 0 \\ 0 & 0 & 0 & \pi & -y & 0 \\ 0 & 0 & 0 & 0 & \pi & -y \\ 0 & 0 & 0 & y & 0 & \pi^3 \end{bmatrix} \sim \begin{bmatrix} 0 & y^2 & -\pi^3 & 0 & 0 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 & 0 & 0 \\ 0 & -\pi^2 & -y & 0 & 0 & 0 \\ \pi^2 & 0 & \pi & -y & 0 & 0 \\ 0 & 0 & 0 & 0 & y^2 & \pi^2 \\ 0 & 0 & 0 & 0 & \pi^3 & -y \end{bmatrix}$$

which is not split, so that the middle term of the AR sequence ending in A_1 is $N_1 \oplus D_1$. This gives the AR sequence $0 \to B_1 \to N_1 \oplus D_1 \to A_1 \to 0$. Taking syzygies gives the AR sequence ending in B_1: $0 \to A_1 \to M_1 \oplus C_1 \to B_1 \to 0$.

The computation for $A_2 \cong (y^2, y\pi^2, \pi^4)R$ is very similar. Let h be the endomorphism of A_2 given by multiplication by π^4/y. Then yh is multiplication by π^4 and πh is multiplication by $-y^2$ on A_2. These both give split sequences: The maps

$$x \mapsto \begin{pmatrix} -x \\ 0 \\ 0 \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

give factorizations of $-y^2$ and π^4, respectively, through R^3. We can compute the middle term of the exact sequence given by pulling back along h, and if it is nonsplit, we will have identified the AR sequence ending in A_2. Since we can write

$$h = \operatorname{Coker}\left(\begin{bmatrix} 0 & 0 & -\pi^3 \\ \pi^2 & 0 & 0 \\ 0 & \pi^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{bmatrix} \right)$$

the middle term is given by the matrix

$$\chi_1 = \begin{bmatrix} \beta_2 & \begin{pmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{pmatrix} \\ 0 & \alpha_2 \end{bmatrix}$$

so the AR sequence ending in A_2 is $0 \to B_2 \to X_1 \to A_2 \to 0$. Taking syzygies gives the AR sequence ending in B_2: $0 \to A_2 \to Y_1 \to B_2 \to 0$.

In computing the rest of the AR sequences, we can use the process of elimination and compute ranks, since R is a domain. First consider the AR sequence ending in C_1. The first term is D_1, which has rank 2, so the middle term has rank 4. Since we already have an arrow $A_1 \to C_1$ in the AR quiver, we know that the middle term has a direct summand isomorphic to A_1. The complement has rank 3, and its first syzygy has rank 2 (since B_1 has rank 2). So the complement is either X_2 or a direct sum of B_1 and a rank-one module which has rank-one first syzygy. If the latter, then the rank-one would be one of M_1, M_2, N_1, or N_2. But we have already computed these AR sequences, and have no arrow from D_1 to any of these. So X_1 is the complement, and the AR sequence ending in C_1 is given by $0 \to D_1 \to A_1 \oplus X_2 \to C_1 \to 0$. Taking syzygies gives $0 \to C_1 \to B_1 \oplus Y_2 \to D_1 \to 0$, the AR sequence ending in D_1.

Next consider the AR sequence ending in C_2. Since C_2 and D_2 have rank 2, the middle term of the AR sequence has rank 4. We know that the AR quiver contains an arrow $M_2 \to C_2$, so M_2 is a summand of the middle term, leaving a rank-three complement with rank-three first syzygy. This complement contains none of the modules we have treated up to now, so must be one of X_1, X_2, Y_1, or Y_2. We know that Y_2 has the wrong rank, and the first syzygy of X_2 (that is, Y_2) also has the wrong rank. It can be checked that the map h on C_2 taking the generators (elements of R^4) to

$$\begin{bmatrix} y\pi^4 \\ -y^2\pi^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -y^2\pi \\ -\pi^4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y^2\pi \\ \pi^4 \\ -y^2\pi^2 \end{bmatrix}, \begin{bmatrix} \pi^5 \\ 0 \\ -y^2\pi \\ -y\pi^4 \end{bmatrix}$$

satisfies $yh = \pi^4$ and $\pi h = -y^2$. Each of these splits a free presentation of C_2. The endomorphism of C_2 given by multiplication by y^2 factors through R^4 by sending the generators (columns of δ_2) to the columns of

$$\begin{bmatrix} 0 & -\pi^2 & 0 & 0 \\ -y\pi^3 & 0 & y\pi^2 & -\pi^4 \\ 0 & 0 & 0 & y\pi^2 \\ 0 & 0 & -\pi^3 & 0 \end{bmatrix}.$$

The endomorphism given by multiplication by π^4 factors through R^4 by sending the generators to the columns of the matrix

$$\begin{bmatrix} 0 & -y\pi & 0 & -y^2 \\ y^2\pi^2 & 0 & -y^2\pi & 0 \\ 0 & 0 & 0 & -y^2\pi \\ 0 & 0 & y\pi^2 & 0 \end{bmatrix}.$$

We can factor h as a pair of maps between free modules

$$h = \operatorname{Coker}\left(\begin{bmatrix} 0 & 0 & 0 & -y \\ 0 & 0 & -y\pi & 0 \\ y^2 & \pi^2 & 0 & -y\pi \\ \pi^3 & -y & \pi^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y\pi & 0 & y \\ -\pi^2 & 0 & y\pi & 0 \\ y & \pi^2 & 0 & 2\pi \\ \pi^3 & -y^2 & -2y\pi^2 & 0 \end{bmatrix} \right)$$

Hence the middle term of the short exact sequence obtained by pulling back via h is presented by the matrix

$$\begin{bmatrix} y & \pi^2 & 0 & \pi & 0 & y\pi & 0 & y \\ \pi^3 & -y^2 & -y\pi^2 & 0 & -\pi^2 & 0 & y\pi & 0 \\ 0 & 0 & y^2 & \pi^2 & y & \pi^2 & 0 & 2\pi \\ 0 & 0 & \pi^3 & -y & \pi^3 & -y^2 & -2y\pi^2 & 0 \\ 0 & 0 & 0 & 0 & y^2 & \pi^2 & 0 & y\pi \\ 0 & 0 & 0 & 0 & \pi^3 & -y & -\pi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & \pi^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi^3 & -y^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} y & \pi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \pi^3 & -y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi^4 & y\pi^3 & y^2 & 0 & 0 & y\pi \\ 0 & 0 & -y^2 & \pi^3 & y\pi & -y & 0 & 0 \\ 0 & 0 & -y\pi^2 & -y^2 & \pi^3 & 0 & -y\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi^2 & -y \\ 0 & 0 & 0 & 0 & 0 & y & 0 & \pi^2 \end{bmatrix}.$$

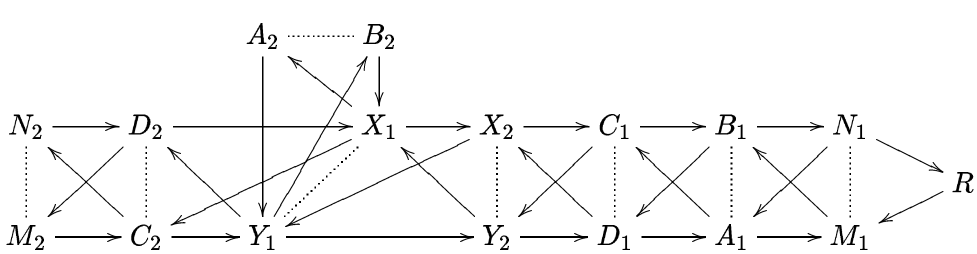
This is the presentation matrix for $M_2 \oplus X_1$, so the AR sequence ending in C_2 is given by $0 \to D_2 \to M_2 \oplus X_1 \to N_2 \to 0$. Taking syzygies gives the AR sequence ending in D_2: $0 \to C_2 \to N_2 \oplus Y_1 \to D_2 \to 0$.

The middle term of the AR sequence ending in X_1 has rank 6, and has summands of B_2 and D_2 from existing arrows in our quiver. The only other rank-two is Y_2. (Besides, we know the complement must have rank 2 and a rank-three syzygy, so must be Y_2.) This gives the two AR sequences $0 \to Y_1 \to B_2 \oplus D_2 \oplus Y_2 \to X_1 \to 0$ and $0 \to X_1 \to A_2 \oplus C_2 \oplus X_2 \to Y_1 \to 0$.

Finally consider the AR sequence ending in X_2. Since X_2 has rank 3 and Y_2 has rank 2, the middle term has rank 5. We already have an arrow $D_1 \to X_2$ and an arrow $X_1 \to X_2$, so the middle term is $D_1 \oplus X_1$, and the AR sequence is

$$0 \to Y_2 \to D_1 \oplus X_1 \to X_2 \to 0.$$

Applying Theorem 2.9 shows that the AR quiver for R is as follows.



(E_8)

The (E_8') singularity. Let $R = V[[y]]/(\pi^3 + y^5)$, with residue field characteristic $p \neq 5$. Once again, the symmetry of this case with the (E_8) singularity implies that R has finite CM type.

References

[1] M. Auslander, *Isolated singularities and the existence of almost split sequences*, Proc. ICRA IV, Lecture Notes in Mathematics, vol. 1178, Springer-Verlag, New York-Berlin, 1986, pp. 194–241.

[2] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer, *Cohen–Macaulay modules on hypersurface singularities II*, Invent. Math. **88** (1987), 165–182.

[3] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, Berlin, 1995.

[4] G.-M. Greuel and H. Knörrer, *Einfache Kurvensingularitäten und torsionfreie Moduln*, Math. Ann. **270** (1985), 417–425.

[5] G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), 229–354.

[6] M. Harada and Y. Sai, *On categories of indecomposable modules I*, Osaka J. Math. **8** (1971), 309–321.

[7] J. Herzog, *Ringe mit nur endlich vielen Isomorphieklassen von maximalen unzerlegbaren Cohen–Macaulay Moduln*, Math. Ann. **233** (1978), 21–34.

[8] H. Knörrer, *Cohen–Macaulay modules on hypersurface singularities I*, Invent. Math. **88** (1987), 153–164.

[9] S. Lang, *Algebra, 3rd. ed.*, Addison–Wesley, 1993.

[10] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Math., vol. 8, Cambridge University Press, Cambridge, 1986.

[11] R. S. Pierce, *Associative Algebras*, Graduate Texts in Math., vol. 88, Springer-Verlag, 1982.

[12] Ø. Solberg, *Hypersurface singularities of finite Cohen–Macaulay type*, Proc. London Math. Soc. **58** (1989), 258–280.

[13] R. Wiegand, *Curve singularities of finite Cohen-Macaulay type*, Ark. Mat. **29** (1991), no. 2, 339–357.

[14] R. Wiegand and S. Wiegand, *Prime ideals and direct-sum decompositions*, to appear in Non-Noetherian Ring Theory, S. Chapman and S. Glaz, eds.

[15] Y. Yoshino, *Brauer–Thrall type theorem for maximal Cohen–Macaulay modules*, J. Math. Soc. Japan **39** (1987), 719–739.

[16] ______, *Cohen–Macaulay modules over Cohen–Macaulay rings*, London Math. Soc. Lect. Notes Ser., vol. 146, Cambridge University Press, 1990.

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