ON A CONJECTURE OF AUSLANDER AND REITEN

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Abstract. In studying Nakayama's 1958 conjecture on rings of infinite dominant dimension, Auslander and Reiten proposed the following generalization: Let Λ be an Artin algebra and M a Λ-generator such that $\operatorname{Ext}^i_\Lambda(M,M)=0$ for all $i \geq 1$; then M is projective. This conjecture makes sense for any ring. We establish Auslander and Reiten's conjecture for excellent Cohen–Macaulay normal domains containing the rational numbers, and slightly more generally.

0. Introduction

The generalized Nakayama conjecture of M. Auslander and I. Reiten is as follows [4]: For an Artin algebra Λ, every indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ. Equivalently, if M is a finitely generated Λ-generator such that $\operatorname{Ext}^i_\Lambda(M,M)=0$ for all $i \geq 1$, then M is projective. This latter formulation makes sense for any ring, and Auslander, S. Ding, and Ø. Solberg [3] widened the context to algebras over commutative local rings.

Conjecture (AR). *Let Λ be a Noetherian ring finite over its center and M a finitely generated left Λ-module such that $\operatorname{Ext}^i_\Lambda(M,\Lambda)=\operatorname{Ext}^i_\Lambda(M,M)=0$ for all $i>0$. Then M is projective.*

In the same paper, Auslander and Reiten proved AR for modules M that are *ultimately closed*, that is, there is some syzygy N of M all of whose indecomposable direct summands already appear in some previous syzygy of M. This includes all modules over rings of finite representation type, all rings Λ such that for some integer n, Λ has only a finite number of indecomposable summands of n^{th} syzygies, and all rings of radical square zero.

Auslander, Ding, and Solberg [3, Proposition 1.9] established AR in case Λ is a quotient of a ring Γ of finite global dimension by a regular sequence. In fact, in this case they prove something much

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stronger: If $\mathrm{Ext}^2_\Lambda(M,M)=0$, then $\mathrm{pd}_\Lambda M<\infty$ [3, Proposition 1.8]. This in turn was generalized by L. Avramov and R.-O. Buchweitz [5, Theorem 4.2]: A finite module M over a (commutative) complete intersection ring R has finite projective dimension if and only if $\mathrm{Ext}^{2i}_R(M,M)=0$ for some $i>0$.

M. Hoshino [9] proved that if R is a symmetric Artin algebra with radical cube zero, then $\mathrm{Ext}^1_R(M,M)=0$ implies that M is free. Huneke, L.M. Şega, and A.N. Vraciu have recently extended this to prove that if R is Gorenstein local with $\mathfrak{m}^3=0$, and if $\mathrm{Ext}^i_R(M,M)=0$ for some $i\geq 1$, then M is free, and have further verified the Auslander-Reiten conjecture for all finitely generated modules M over Artinian commutative local rings $(R,\mathfrak{m})$ such that $\mathfrak{m}^2M=0$ [11]. In particular, this verifies the Auslander-Reiten conjecture for commutative local rings with $\mathfrak{m}^3=0$.

The assumption that Λ be finite over its center is essential, given a counterexample due to R. Schultz [15].

Our main theorem establishes the AR conjecture for a class of commutative Cohen–Macaulay rings and well-behaved modules. Moreover, our result is effective; we can specify how many Ext are needed to vanish to give the conclusion of AR.

Main Theorem. *Let R be a Cohen–Macaulay ring which is a quotient of a locally excellent ring S of dimension d by a locally regular sequence. Assume that S is locally a complete intersection ring in codimension one, and further assume either that S is Gorenstein, or that S contains the field of rational numbers. Let M be a finitely generated R-module of constant rank such that*

$$\begin{aligned} &Ext^i_R(M,M)=0 \quad \text{for } i=1,\dots,d, \text{ and}\\ &Ext^i_R(M,R)=0 \quad \text{for } i=1,\dots,2d+1. \end{aligned} \tag{1}$$

Then M is projective.

The restriction imposed on R by assuming that S be locally complete intersection in codimension one is equivalent to assuming that R is a quotient by a regular sequence of some normal domain T, by [10, Theorem 3.1]. However, replacing S by T according to the construction in [10] would increase d, the number of Ext required to vanish. In any case, this observation gives the following corollary.

Theorem 0.1. *Let R be a Cohen–Macaulay ring which is a quotient of a locally excellent ring S of dimension d by a locally regular sequence. Assume that S is locally a complete intersection ring* *in codimension one, and further assume either that S is Gorenstein, or that S contains the field of rational numbers. Then the AR conjecture holds for all finitely generated R-modules, that is, if* $\mathrm{Ext}^i_R(M,R) = \mathrm{Ext}^i_R(M,M) = 0$ *for all $i > 0$, then M is projective.*

Not every zero-dimensional ring R is a factor of a ring S as in the theorem, since not all Artinian local rings can be smoothed. For example, Anthony Iarrobino has pointed out that the easiest such example is a polynomial ring in four variables modulo an ideal generated by seven general quadrics (note, however, that the cube of the maximal ideal of such a ring is zero, so this case is covered by [11]). For other examples of non-smoothable rings, see Mumford [14].

In the next section we prove some preliminary lemmas, and then prove the main result. This requires extra work regarding the trace of a module. Since we could not find a satisfactory reference for what we needed, we include basic facts concerning the trace in an appendix.

Throughout the following, all rings are Noetherian and all modules finitely generated. For an R-module M, we define the *dual* of M by $M^* = \mathrm{Hom}_R(M,R)$. There is a natural homomorphism $\theta_M : M \longrightarrow M^{**}$ defined by sending $x \in M$ to "evaluation at x". We say that M is *torsion-free* if θ_M is injective, and *reflexive* if θ_M is an isomorphism. It is known (cf. [2, Theorem 2.17], for example) that M is torsion-free if and only if M is a first syzygy, and reflexive if and only if M is a second syzygy. We will say that a torsion-free R-module M *has constant rank* if M is locally free of constant rank at the minimal primes of R. This is equivalent to $K \otimes_R M$ being a free K-module, where K is the total quotient ring of R obtained by inverting all nonzerodivisors.

1. Proof of the Main Theorem

We begin by observing that the vanishing of Ext and the projectivity of M are both local questions, so that in proving our main theorem we may assume that both S and R are local. Furthermore, since S is assumed to be excellent we can (and do) complete S at its maximal ideal without loss of generality.

Next we point out the following consequence of the lifting criterion of Auslander, Ding, and Solberg [3, Proposition 1.6].

Lemma 1.1. *Let S be a complete local ring, $x \in S$ a nonunit nonzerodivisor, and $R = S/(x)$. Assume that there exists $t \geq 2$ such that for any S-module N,* $\mathrm{Ext}^i_S(N,N) = \mathrm{Ext}^i_S(N,S) = 0$ *for*

$i = 1, \dots, t$ implies that N is free. Then for any R-module M, $\operatorname{Ext}^i_R(M, M) = \operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, t$ implies that M is free. Furthermore, if AR holds for S-modules then it holds for R-modules.

Proof. Let M be an R-module such that $\operatorname{Ext}^i_R(M, M) = \operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, t$. Then in particular $\operatorname{Ext}^2_R(M, M) = 0$, and so by [3, Proposition 1.6] there exists an S-module N on which x is a nonzerodivisor and such that $R \otimes_S N \cong M$. Apply $\operatorname{Hom}_S(-, N)$ to the short exact sequence $0 \longrightarrow N \longrightarrow N \longrightarrow M \longrightarrow 0$ and use the fact that $\operatorname{Ext}^{i+1}_S(M, N) \cong \operatorname{Ext}^i_R(M, M) = 0$ for $i = 1, \dots, t$ to see that multiplication by x is surjective on $\operatorname{Ext}^i_S(N, N)$ for $i = 1, \dots, t$. Then Nakayama's Lemma implies that $\operatorname{Ext}^i_S(N, N) = 0$ for $i = 1, \dots, t$. The same argument, applying $\operatorname{Hom}_S(-, S)$ and observing that $\operatorname{Ext}^{i+1}_S(M, S) \cong \operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, t$, shows that $\operatorname{Ext}^i_S(N, S) = 0$ for $i = 1, \dots, t$ as well. Since this forces N to be S-free, M is R-free.

Finally, repeating the argument with "all $i \geq 1$" in place of "$i = 1, \dots t$" gives the last statement. □

With Lemma 1.1 in mind, we now focus on the case $R = S$ in our main theorem. Indeed, if $\dim(S) \leq 1$, then S is locally a complete intersection ring by hypothesis, and hence R is as well. By [3, Proposition 1.9], then, AR holds for R-modules. So we may assume that $R = S$, and in particular we take $d = \dim R$. Our next goal is to modify the module M.

Lemma 1.2. [4, Lemma 1.4] *In proving the Main Theorem, we may replace M by $\operatorname{syz}^n_R(M)$, where $n = \max\{2, d+1\}$, and assume that M is reflexive and that $\operatorname{Ext}^i_R(M^*, R) = 0$ for $i = 1, \dots, d$. In proving AR, we may replace M by any syzygy module $\operatorname{syz}^t_R(M)$.*

Proof. Put $N = \operatorname{syz}^n_R(M)$. It is a straightforward computation with the long exact sequences of Ext to show that if $\operatorname{Ext}^i_R(M, M) = 0$ for $i = 1, \dots, d$ and $\operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \dots, 2d+1$, then $\operatorname{Ext}^i_R(N, N) = 0$ for $i = 1, \dots, d$ and $\operatorname{Ext}^i_R(N, R) = 0$ for $i = 1, \dots, d$. Assume, then, that we have shown that N is free. Then since $\operatorname{Ext}^n_R(M, R) = 0$, the n-fold extension of M by N consisting of the free modules in the resolution of M must split, so M is free as well. This proves the last statement.

To prove that N is reflexive and $\operatorname{Ext}^i_R(N^*, R) = 0$ for $i = 1, \dots, d$, one shows by induction on t that $\operatorname{Ext}^i_R((\operatorname{syz}^t_R(M))^*, R) = 0$ for $i = 1, \dots, t$. For the base case $t = 2$, observe that since

$\operatorname{Ext}^i_R(M,R)=0$ for $i=1,2$, the dual of the exact sequence

$$(*) \qquad 0\longrightarrow \operatorname{syz}^2_R(M)\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0,$$

where F_1 and F_0 are free modules, is still exact. Dualizing again gives (*) back, so $N=\operatorname{syz}^2_R(M)$ is reflexive and satisfies $\operatorname{Ext}^i_R(N^*,R)=0$ for $i=1,2$. For the inductive step, dimension-shifting shows that if $\operatorname{Ext}^i_R(M^*,R)=0$ for $i=1,\dots,t-1$, then $\operatorname{Ext}^i_R((\operatorname{syz}^1_R(M))^*,R)=0$ for $i=2,\dots,t$, and the same argument as above shows that $\operatorname{Ext}^1_R((\operatorname{syz}^1_R(M))^*,R)=0$. □

It is worth noting that if R is a Cohen–Macaulay (CM) ring, then $\operatorname{syz}^d_R(M)$ is a maximal Cohen–Macaulay (MCM) module for any M. Also, the replacement in Lemma 1.2 has consequences for the assumptions (1) in the main theorem: If $\operatorname{Ext}^i_R(M,R)=0$ for $i=1,\dots,t$, then $\operatorname{Ext}^i_R(\operatorname{syz}^1_R(M),R)=0$ for $i=1,\dots,t-1$. This observation combines with Lemmas 1.1 and 1.2 to reduce the proof of our main theorem to the following:

Theorem 1.3. *Let $(R,\mathfrak{m})$ be a complete local CM ring of dimension d which is a complete intersection in codimension one. Assume either that R is Gorenstein, or that R contains $\mathbb{Q}$. Let M be a MCM R-module of constant rank such that for $i=1,\dots,d$,*

$$(2) \qquad \begin{aligned} &\operatorname{Ext}^i_R(M,M)=0,\\ &\operatorname{Ext}^i_R(M,R)=0, \quad \textit{and}\\ &\operatorname{Ext}^i_R(M^*,R)=0. \end{aligned}$$

Then M is free.

We postpone the proof of Theorem 1.3 to the end of this section, and establish some preparatory results.

By Cohen's structure theorem, the complete local ring R is a homomorphic image of a regular local ring, and so has a canonical module ω. Since R is complete intersection in codimension one, it is in particular Gorenstein at the associated primes, and so ω has constant rank. Hence ω is isomorphic to an ideal of R. For a MCM R-module N, we write $N^\vee$ for the canonical dual $\operatorname{Hom}_R(N,\omega)$.

We next apply a result found in [6, Corollary B4] (see also [8, Lemma 2.1]):

Proposition 1.4. *Let R be a CM local ring with a canonical module ω and let N be a MCM R-module. If $\operatorname{Ext}^i_R(N,R)=0$ for $i=1,\dots,\dim R$ then $\omega\otimes_R N\cong (N^*)^\vee$ is a MCM R-module.*

Applied to our current context, this gives the following fact.

Corollary 1.5. *Under our assumptions (2) in Theorem 1.3, both $\omega\otimes_R M$ and $\omega\otimes_R M^*$ are MCM R-modules.*

We will also show that the triple tensor product $\omega\otimes_R M^*\otimes_R M$ is MCM, but for this we use the following lemma. It requires that we add one further assumption to (2): that the module M in question has constant rank.

Lemma 1.6. *Let $(R,\mathfrak{m},k)$ be a CM local ring with canonical ideal ω, and let N be a MCM R-module of constant rank. Assume that $\operatorname{Hom}_R(N,N)$ is also a MCM R-module, and that for some maximal regular sequence $\underline{x}$, we have*

$$\operatorname{Hom}_R(N,N)\otimes_R R/(\underline{x})\cong \operatorname{Hom}_{R/(\underline{x})}(N/\underline{x}N,N/\underline{x}N).$$

Then $\underline{x}$ is a regular sequence on $N\otimes_R N^\vee$. In particular, $N\otimes_R N^\vee$ is MCM.

Proof. We indicate reduction modulo $\underline{x}$ by an overline, and use $\lambda(-)$ for the length of a module. We also continue to use $-^\vee$ for $\operatorname{Hom}_{\overline{R}}(-,\overline{\omega})$ without fear of confusion. Since $\overline{\omega}\cong E_{\overline{R}}(k)$, the injective hull of the residue field of $\overline{R}$, we have $\lambda(M^\vee)=\lambda(M)$ for all $\overline{R}$-modules M.

First, a short computation using Hom-Tensor adjointness:

$$\begin{aligned}(\overline{N}\otimes_{\overline{R}}\overline{N}^\vee)^\vee &= \operatorname{Hom}(\overline{N}\otimes_{\overline{R}}\overline{N}^\vee,\overline{\omega})\\ &\cong \operatorname{Hom}_{\overline{R}}(\overline{N},\overline{N}^{\vee\vee})\\ &\cong \operatorname{Hom}_{\overline{R}}(\overline{N},\overline{N})\end{aligned}$$

In particular, this implies that $\lambda(\overline{N}\otimes_{\overline{R}}\overline{N}^\vee)=\lambda((\overline{N}\otimes_{\overline{R}}\overline{N}^\vee)^\vee=\lambda(\operatorname{Hom}_{\overline{R}}(\overline{N},\overline{N}))$. Since $\overline{N\otimes_R N^\vee}=\overline{N}\otimes_{\overline{R}}\overline{N}^\vee$, our hypothesis yields $\lambda(\overline{N\otimes_R N^\vee})=\lambda(\overline{\operatorname{Hom}_R(N,N)})$. Finally, we compute, using the fact that N, $N \otimes_R N^\vee$, and $\mathrm{Hom}_R(N,N)$ all have constant rank:

$$\begin{aligned}\lambda(\overline{N \otimes_R N^\vee}) &= \lambda(\overline{\mathrm{Hom}_R(N,N)})\\ &= e(\underline{x}, \mathrm{Hom}_R(N,N))\\ &= \mathrm{rank}(\mathrm{Hom}_R(N,N))e(\underline{x},R)\\ &= \mathrm{rank}(N)^2 e(\underline{x},R)\\ &= \mathrm{rank}(N \otimes_R N^\vee)e(\underline{x},R)\\ &= e(\underline{x}, N \otimes_R N^\vee)\end{aligned}$$

Here $e(\underline{x},\ )$ denotes the multiplicity of the ideal $(\underline{x})$ on the module. The second equality follows since we have assumed that $\mathrm{Hom}_R(N,N)$ is also a MCM R-module. The equality of the first and last items implies that $N \otimes_R N^\vee$ is MCM by [7, 4.6.11]. $\square$

Proposition 1.7. *Let $(R,\mathfrak{m})$ be a CM local ring with canonical ideal ω and let M be a reflexive R-module of constant rank such that $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^i_R(M^*,R) = 0$ for $i = 1,\dots,d = \dim R$. Then $\omega \otimes_R M^* \otimes_R M$ is a MCM R-module.*

Proof. We will take $N = M$ in Lemma 1.6. By Proposition 1.4, $M^\vee \cong \omega \otimes_R M^*$, so we need only show that $\mathrm{Hom}_R(M,M)$ cuts down correctly. Induction on the length of a regular sequence $\underline{x}$, using the vanishing of $\mathrm{Ext}^i_R(M,M)$, then proves that $\underline{x}$ is also regular on $\mathrm{Hom}_R(M,M)$ and that $\mathrm{Hom}_R(M,M) \otimes_R R/(\underline{x}) \cong \mathrm{Hom}_{R/(\underline{x})}(M/\underline{x}M, M/\underline{x}M)$, finishing the proof. $\square$

Proposition 1.8. *In addition to the assumptions (2) of Theorem 1.3, suppose also that M has constant rank. Then $\omega \otimes_R M^* \otimes_R M$ is a MCM R-module. Furthermore, the natural homomorphism*

$$1 \otimes \alpha : \omega \otimes_R M^* \otimes_R M \longrightarrow \omega \otimes_R \mathrm{Hom}_R(M,M),$$

where α is defined by $\alpha(f \otimes x)(y) = f(y) \cdot x$, is injective.

Proof. The first statement follows immediately from Proposition 1.7. For the second, pass to the total quotient ring K of R. Since R is generically Gorenstein, $\omega \otimes_R K \cong K$, and since M has a rank, $M \otimes_R K$ is a free K-module. Since α is an isomorphism when M is free, the kernel of $1 \otimes \alpha$ must be torsion. But $\omega \otimes_R M^* \otimes_R M$ is MCM, and so torsion-free. Hence the kernel of $1 \otimes \alpha$ is zero. $\square$

We return to the assumptions of Theorem 1.3: $(R, \mathfrak{m}, k)$ is a complete local CM ring with a canonical ideal ω, and M is a torsion-free R-module of constant rank, satisfying

$$\begin{aligned} &\operatorname{Ext}_R^i(M, M) = 0, \\ &\operatorname{Ext}_R^i(M, R) = 0, \quad \text{and} \\ &\operatorname{Ext}_R^i(M^*, R) = 0, \quad \text{for } i = 1, \ldots, d = \dim R. \end{aligned} \tag{3}$$

We also assume that R is locally a complete intersection ring in codimension one. As we observed above, this implies by the work of Auslander, Ding, and Solberg that M is locally free in codimension one. We therefore assume $d \geq 2$. The following lemma is standard. (See [13, Theorems 16.6, 16.7].)

Lemma 1.9. *Let $(R, \mathfrak{m}, k)$ be a CM local ring of dimension at least 2. Let X be a MCM R-module and L a module of finite length over R. Then $\operatorname{Ext}_R^1(L, X) = 0$.*

Recall from Proposition 1.8 that under the assumptions (3), the homomorphism $1 \otimes \alpha : \omega \otimes_R M^* \otimes_R M \longrightarrow \omega \otimes_R \operatorname{Hom}_R(M, M)$ is injective.

Lemma 1.10. *If M is locally free on the punctured spectrum, then the homomorphism $1 \otimes \alpha$ is a split monomorphism with cokernel of finite length.*

Proof. We have the following exact sequence:

$$0 \longrightarrow \omega \otimes_R M^* \otimes_R M \xrightarrow{1 \otimes \alpha} \omega \otimes_R \operatorname{Hom}_R(M, M) \longrightarrow C \longrightarrow 0. \tag{4}$$

Since M is locally free on the punctured spectrum, $1 \otimes \alpha$ is an isomorphism when localized at any nonmaximal prime of R, which forces C to have finite length. Since $\omega \otimes_R M^* \otimes_R M$ is MCM by Proposition 1.8, $\operatorname{Ext}_R^1(C, \omega \otimes_R M^* \otimes_R M) = 0$, and so (4) splits. □

Proof of Theorem 1.3. We will proceed by induction on $d = \dim R$. As mentioned above, the case $d = 1$ follows from [3, Proposition 1.9], so we may assume $d \geq 2$, and that the statement is true for all modules over CM local rings matching our hypotheses (3) and having dimension less than that of R. In particular, we may assume that M is locally free on the punctured spectrum. Also, we may assume that M is indecomposable.

First assume that R is Gorenstein. Then $\alpha : M^* \otimes_R M \longrightarrow \operatorname{Hom}_R(M, M)$ must be a split monomorphism with cokernel of finite length, by Lemma 1.10. Since $\operatorname{Hom}_R(M, M)$ is torsion-free, this implies α is an isomorphism, and hence that M is free.

Next assume that R is not necessarily Gorenstein, but contains the rationals. Consider the following diagram involving the trace homomorphism (see Appendix A).

$$\begin{array}{ccc} \omega \otimes_R M^* \otimes_R M & \xrightarrow{1\otimes\alpha} & \omega \otimes_R \operatorname{Hom}_R(M,M) \\ & \searrow^{1\otimes \mathrm{ev}} & \downarrow{\scriptstyle 1\otimes\mathrm{tr}} \\ & & \omega \otimes_R R \end{array}$$

By Lemma A.6, the diagram commutes. Furthermore, by Lemma 1.10, $1 \otimes \alpha$ is a split monomorphism with finite-length cokernel C, so $\omega \otimes_R \operatorname{Hom}_R(M,M)$ has C as a direct summand and $1 \otimes \alpha$ is surjective onto the complement. Since ω is torsion-free, $1 \otimes \mathrm{tr}$ must kill C.

As R contains $\mathbb{Q}$, $\operatorname{rank} M$ is invertible and so tr is surjective by Corollary A.5. It follows that the composition $1 \otimes \mathrm{tr}\,\alpha$ is surjective, so that $1 \otimes \mathrm{ev}$ is as well. In other words, the evaluation map $M^* \otimes_R M \longrightarrow R$ induces a surjection when tensored with ω. By Nakayama's Lemma, then, the evaluation map is surjective, and it follows that M has a free direct summand. Since M is indecomposable, M is free. □

Appendix A. The Trace of a Module

In this section we give a general description of the trace of a module. Our treatment is intrinsic to the module, and it satisfies the usual properties of a trace defined for torsion-free modules over a normal domain. We include full proofs for convenience.

Throughout this section, let R be a Noetherian ring with total quotient ring K; that is, K is obtained from R by inverting all nonzerodivisors. Let M be a torsion-free R-module. The *trace of* M will be a certain homomorphism $\mathrm{tr} : \operatorname{Hom}_R(M,M) \longrightarrow R$. To define the trace, let

$$\alpha : M^* \otimes_R M \longrightarrow \operatorname{Hom}_R(M,M)$$

be the natural homomorphism defined by $\alpha(f \otimes x)(y) = f(y)\cdot$. Note that dualizing α gives a homomorphism α^* from $\operatorname{Hom}_R(M,M)^* = \operatorname{Hom}_R(\operatorname{Hom}_R(M,M),R)$ to $(M^* \otimes_R M)^* \cong \operatorname{Hom}_R(M^*,M^*)$. It is known (see [12], for example) that α is an isomorphism if and only if M is free.

Definition A.1. *Assume that* $\alpha^* : \operatorname{Hom}_R(M,M)^* \longrightarrow \operatorname{Hom}_R(M^*,M^*)$ *is an isomorphism. The* trace of M *is defined by* $\mathrm{tr} = (\alpha^*)^{-1}(1_{M^*})$. *We say in this case that* M has a trace.

Observe that the target of α^* is $(M^* \otimes_R M)^*$, which we have used Hom-Tensor adjointness to identify with $\operatorname{Hom}_R(M^*, M^*)$. Under this identification, the identity map $M^* \longrightarrow M^*$ corresponds to the evaluation map $\operatorname{ev} : M^* \otimes_R M \longrightarrow R$ defined by $\operatorname{ev}(f \otimes x) = f(x)$. To see this, recall that the Hom-Tensor morphism $\Phi_{ABC} : \operatorname{Hom}(A \otimes B, C) \longrightarrow \operatorname{Hom}(A, \operatorname{Hom}(B, C))$ is defined by $[\Phi_{ABC}(f)(a)](b) = f(a \otimes b)$ for $a \in A$, $b \in B$. Taking $A = M^*$, $B = M$, $C = R$, we see that for $x \in M$ and $f \in M^*$, $[\Phi_{M^*MR}(\operatorname{ev})(f)](x) = \operatorname{ev}(f \otimes x) = f(x)$. So $\Phi_{M^*MR}(\operatorname{ev})$ is the map $M^* \longrightarrow M^*$ taking f to f. In particular, we could also define the trace by $\operatorname{tr} = (\alpha^*)^{-1}(\operatorname{ev})$.

Our first proposition generalizes the standard fact that a torsion-free module over a normal domain has a trace.

Proposition A.2. *If $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for all primes $\mathfrak{p}$ of height one in R, and R satisfies Serre's condition (S_2), then M has a trace.*

Proof. We must show that $\alpha^* : \operatorname{Hom}_R(M, M)^* \longrightarrow \operatorname{Hom}_R(M^*, M^*)$ is an isomorphism. Let $L = \ker(\alpha)$, $I = \operatorname{im}(\alpha)$, $C = \operatorname{coker}(\alpha)$. Then dualizing α gives two exact sequences:

$$0 \longrightarrow I^* \longrightarrow \operatorname{Hom}_R(M^*, M^*) \longrightarrow L^*$$

$$0 \longrightarrow C^* \longrightarrow (\operatorname{Hom}_R(M, M)^* \longrightarrow I^* \longrightarrow \operatorname{Ext}^1_R(C, R)$$

Since α is an isomorphism at all minimal primes of R, the annihilator of L is not contained in any minimal prime. Hence L is a torsion module, and so $L^* = 0$.

Since, further, α is an isomorphism at all primes of height one in R, the annihilator of C is not contained in any height-one prime. By the assumption that R satisfies condition (S_2), then, $\operatorname{grade}(\operatorname{Ann} C) \geq 2$, so $C^* = \operatorname{Ext}^1_R(C, R) = 0$. This shows that α^* is an isomorphism. □

Lemma A.3. *For $f \in \operatorname{Hom}_R(R^n, R^n)$, $\operatorname{tr}(f)$ is the sum of the diagonal entries of a matrix representing f.*

Proof. Since R^n is free, α is an isomorphism already, and of course α^* is as well. Write $f = \alpha(\sum_{i=1}^n a_{ij} g_j \otimes e_i)$, where e_i and g_i are the canonical bases for R^n and its dual, respectively. Then since $g_j(e_i) = \delta_{ij}$, we see that

$$\operatorname{tr}(f) = \operatorname{ev}(\sum_{i=1}^n a_{ij} g_j \otimes e_i) = \sum_{1 \leq i,j \leq n} a_{ij} g_j(e_i) = \sum_{1 \leq j \leq n} a_{jj},$$

as desired. □

Recall that the torsion-free R-module M is said to have constant rank n if $K \otimes_R M$ is a free K-module of rank n. If this is the case, we fix a basis $\{e_1, \ldots, e_n\}$ for $K \otimes_R M$, and let $\{g_1, \ldots, g_n\}$ be the dual basis, so that $g_i(e_j) = \delta_{ij}$.

Lemma A.4. *Assume that M is a torsion-free R-module of constant rank and that M has a trace. Then for any $f \in M^*$ and $x \in M$, we have $x = \sum_{i=1}^n g_i(x)e_i$ and $\operatorname{tr}(f) = \sum_{i=1}^n g_i(\widehat{f}(e_i))$, where $\widehat{f} = K \otimes_R f$.*

Proof. Since M is torsion-free, it embeds into a free R-module and so the homomorphism $M \longrightarrow K \otimes_R M$ is injective. Considering x as an element of $K \otimes_R M$, write $x = \sum_{j=1}^n a_j e_j$, where the a_j are elements of K. Then a short computation using the definition of the g_i shows that $\sum_{i=1}^n g_i(x)e_i = x$. For the other assertion, pass to the total quotient ring K. Since $K \otimes_R M$ is free, Lemma A.3 implies that the trace of $\widehat{f}$ is the sum of the diagonal elements of a matrix (a_{ij}) representing $\widehat{f}$. Since $g_i(\widehat{f}(e_i)) = a_{ii}$, the statement follows. □

Corollary A.5. *Assume that M is a torsion-free module of constant rank and has a trace. If* $\operatorname{rank}(M)$ *is invertible in R, then* tr *is surjective from* $\operatorname{Hom}_R(M, M)$ *to R.*

Lemma A.6. *Assume that M is a torsion-free of constant rank and that M has a trace. Then we have* $\operatorname{tr} \alpha = \operatorname{ev}$ *as homomorphisms from $M^* \otimes_R M$ to R.*

Proof. For any $f \in M^*$ and $x \in M$, a straightforward computation using Lemma A.4 shows that $f(x) = \operatorname{tr}(\alpha(f \otimes x))$. □

References

1. M. Auslander, *Modules over unramified regular local rings*, Ill. J. Math **5** (1961), 631–647.
2. M. Auslander and M. Bridger, *Stable Module Theory*, Memoirs Amer. Math. Soc., no. 94, Amer. Math. Soc., Providence, R.I., 1969.
3. M. Auslander, S. Ding, and Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), 273–317.
4. M. Auslander and I. Reiten, *On a generalized version of the Nakayama Conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74.
5. L.L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318.
6. L.L. Avramov, R.-O. Buchweitz, and L.M. Şega, *Extensions of a dualizing complex by its ring: Commutative versions of a conjecture of Tachikawa*, Preprint, 2002.

7. W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Stud. in Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993.
8. D. Hanes and C. Huneke, *Some criteria for the Gorenstein property*, to appear, J. Pure Appl. Algebra.
9. M. Hoshino, *Modules without self-extensions and Nakayama's conjecture*, Arch. Math. **43** (1984), 494–500.
10. C. Huneke and B. Ulrich, *Divisor class groups and deformations*, Amer. J. Math. **107** (1985), 1265–1303.
11. C. Huneke, L.M. Şega, and A.N. Vraciu, *Vanishing of Ext and Tor over Cohen–Macaulay local rings*, Preprint, 2003.
12. G. J. Leuschke, *Gorenstein modules, finite index, and finite Cohen–Macaulay type*, Comm. Algebra **30** (2002), no. 4, 2023–2035.
13. H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8** (1986) Cambridge University Press.
14. D. Mumford, *Pathologies IV*, Amer. J. Math. **97** (1975), no. 3, 847-849.
15. R. Schultz, *A nonprojective module without self-extensions*, Arch. Math. (Basel) **62** (1994), no. 6, 497-500.
16. H. Tachikawa, *Quasi-Frobenius rings and generalizations*, QF-3 and QF-1 rings, Lect. Notes. Math., vol. 351, Springer, Berlin, 1973.

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