THE BANACH-TARSKI PARADOX

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The usual statement of the Banach-Tarski Paradox is this: a solid ball can be partitioned into finitely many pieces and those pieces rearranged *by rigid motions*, that is, by translations and rotations, to form two solid balls the same size as the first.

Another, more impressive version: A pea can be cut up and reassembled into a ball the size of the sun.

My goal here today is to state the BTP carefully, give you an idea of how it is proven, and (along the way) explain why this geometric statement of interest to analysts and why an algebraist is talking about it. It turns out that the statement is essentially equivalent to the Axiom of Choice, but that it has interesting analytical consequences, and the main idea of the proof is algebraic.

This entire talk is essentially taken from Stan Wagon's excellent book "The Banach-Tarski Paradox", which has a great deal more about the history, the proof, consequences, and "philosophical" implications of the theorem.

First let's consider a problem in the plane that was solved in the early 1800s. Say two polygons are *congruent by dissection* if one of them can be decomposed into finitely many polygons which can be rearranged using *isometries* (and ignoring boundaries) to form the other one.

Clearly any two polygons that are congruent by dissection have the same area.

Theorem 1 (Wallace 1807, Bolyai-Gerwien 1833)**.** *The converse is true.*

To prove this, just show that every triangle is c. by d. to a rectangle, then any rectangle is c. by d. to a square, and finally that any finite set of squares is c. by d. to a single square. It's a good exercise.

The analogous problem for polyhedra in $\mathbb{R}^n$ for $n \geq 3$ is not nearly as simple. In fact, Gauss pointed out (in a letter to Gerling, around 1815) that all the proofs of the volume of a tetrahedron rely on the "method of exhaustion", basically a limiting argument, and he asked for a geometric proof by dissection. Hilbert's 3rd problem was whether or not a regular tetrahedron in $\mathbb{R}^3$ is

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congruent by dissection (into polyhedra) with a cube. Max Dehn gave a proof in 1900 that no such dissection exists.

Note that in the definition of "congruence by dissection," we're appealing to the group of *isometries* of $\mathbb{R}^2$, that is, the set of rigid motions of the plane. If we were to change the group of motions that we allow, we would of course get different results. If we allow scalar multiplication as well, then we could say that *any* two polygons are "congruent."

This suggests that we consider *group actions on metric spaces*. A group G *acts on* a set X if for each $g \in G$ there is a bijection $g : X \to X$, such that $g(h(x)) = (gh)(x)$ and $1(x) = x$ for all $g, h \in G$ and $x \in X$.

Examples of groups acting on $\mathbb{R}^n$:

- $GL(n)$ is the full linear group,
- $\mathbb{A}(n)$ is the *affine group* of linear maps and translations,
- $E(n)$ is the group of all isometries,
- $O(n)$ is the *orthogonal group*, consisting of those isometries leaving the origin fixed, and
- $SO(n)$ is the *special orthogonal group*, which leaves the origin fixed and preserves orientation.

In particular, $SO(3)$ just consists of the rotations of $\mathbb{R}^3$ around the origin.

Definition 2. Let G be a group acting on a set X. Say that a subset $Z \subseteq X$ is G-*paradoxical* if there are pairwise-disjoint subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of Z and group elements $g_1, \dots, g_n, h_1, \dots, h_m$ such that

$$Z = \bigcup_i g_i(A_i) \qquad \text{and} \qquad Z = \bigcup_j h_j(B_j)\,.$$

Roughly speaking, this means that Z has two disjoint families of subsets, each of which can be rearranged to give Z. It will turn out that in fact the subsets can be chosen so that $\{g_i(A)\}$, $\{h_j(B_j)\}$, and $\{A_i\} \cup \{B_j\}$ are each partitions of Z.

The Banach-Tarski Paradox (1924): Any ball in $\mathbb{R}^3$ is $E(3)$-paradoxical.

Note that this says nothing about the pieces – they need not be connected or anything (e.g., could be "the set of points with rational coordinates").

A natural source of group actions is to let a group act *on itself* by (say) left-multiplication. This turns out to be the key to the whole problem.

Theorem 3. *Let F be the free (non-Abelian) group on two generators. Then F is F-paradoxical.*

Idea of proof. Let σ and τ be generators for F, so F consists of (reduced) words in $\sigma, \sigma^{-1}, \tau$, and τ^{-1}. Let H_σ be the subset of words beginning with σ, and similarly $H_{\sigma^{-1}}, H_\tau$, and $H_{\tau^{-1}}$. Then

$$F = \{1\} \cup H_\sigma \cup H_{\sigma^{-1}} \cup H_\tau \cup H_{\tau^{-1}}$$

and these are pairwise disjoint. Furthermore

$$F = H_\sigma \cup \sigma H_{\sigma^{-1}} \qquad \text{and} \qquad F = H_\tau \cup \tau H_{\tau^{-1}}$$

so this gives the desired partition. (This can be jazzed up to use only 4 pieces, one including the identity, instead of 5.) □

In fact, this kind of "auto-paradoxicality" is the root of the whole issue.

Theorem 4. *If G is G-paradoxical and acts without fixed points on a set X, then X is G-paradoxical. Hence X is F-paradoxical whenever a free group of rank 2 acts on it.*

This theorem requires the Axiom of Choice. Here's a reminder about that.

Axiom of Choice: *Given any set X of nonempty sets, there is a set C containing one element from each member of X.*

The A of C is known to be independent of Zermelo-Fraenkel set theory (Gödel and Cohen). It's equivalent with the well-ordering principle and Zorn's Lemma. All in all, a strange character, even though it seems so natural. Oddly, its negation leads to equally strange statements, such as the existence of two incomparable cardinals.

Proof of Theorem 4. Suppose we have disjoint $A_i, B_j \subseteq G$ and $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ with $\bigcup g_i(A_i) = G$ and $\bigcup_j h_j(B_j) = G$.

Recall that an *orbit* of the action of G on X is a set of the form

$$G_x = \{g(x) \mid g \in G\}$$

for some $x \in X$. Using Choice, there is a set C containing one point from each orbit. Then $\{g(C) \mid g \in G\}$ is a partition of X; every point is in some orbit, and they're disjoint because there are no fixed points.

Let

$$A_i^* = \{g(C) \mid g \in A_i\} \qquad \text{and} \qquad B_j^* = \{g(C) \mid g \in B_j\}.$$

Then the A_i^* and B_j^* cover X, and are disjoint (since the A_i and B_j are). Since $\bigcup g_i(A_i) = G$ and $\bigcup_j h_j(B_j) = G$, we have $\bigcup g_i(A_i^*) = X$ and $\bigcup_j h_j(B_j^*) = X$. □

The last main ingredient is to see that $E(3)$ contains a free group of rank 2. In fact there's on in $SO(3)$.

Proposition 5. *The elements*

$$\phi = \begin{pmatrix} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

generate a free (non-Abelian) subgroup of $SO(3)$, and so of $E(3)$.

Note that ϕ and ρ are rotations by $\arccos 1/3$ around the z-axis and x-axis, respectively.

So we're done, right? Not quite: The action of the rotations on the 2-sphere has fixed points. Luckily, there are only countably many (since each rotation has two, and there are only countably many rotations in our group). Thus we've proven the

Hausdorff Paradox (1914): *There is a countable set D of the unit sphere S^2 in $\mathbb{R}^3$ such that $S^2 \setminus D$ is $SO(3)$-paradoxical (and so $E(3)$-paradoxical).*

Corollary: *There is no finitely additive, rotation-invariant measure on the set of subsets of S^2 with total measure 1.*

Sketch of Proof of Corollary. Let μ be such a measure. Then it follows from the HP that $\mu(S^2 \setminus D) = 0$ since it would have to have twice its own measure. So it's enough to show that $\mu(D) = 0$. Choose any line L through the origin missing D. Then there are uncountably many rotations around L such that D and its image are disjoint. If σ is one, then $\mu(S^2) \geq \mu(D \cup \sigma(D)) = 2\mu(D)$, so $\mu(D) \leq \mu(S^2)/2$. Repeat to get $\mu(D) \leq \epsilon$. □

To get from HP to BTP, we need one more concept.

Definition: Suppose G acts on X. Two subsets $A, B \subset X$ are called G-*equidecomposable* if A and B can be partitioned into the same number of pieces, which are pairwise G-congruent. Formally,

$$A = \bigcup_{i=1}^{n} A_i \qquad \text{and} \qquad B = \bigcup_{i=1}^{n} B_i$$

with $A_i \cap A_j = \emptyset = B_i \cap B_j$, and there exist g_i so that $g_i(A_i) = B_i$.

Some facts that are not too hard, but are time-consuming:

- G-equidecomposability is an equivalence relation on subsets of X.
- X is G-paradoxical iff X contains disjoint A and B so that both A and B are G-equidecomposable with X.

- Suppose A, B are G-equidecomposable. Then A is G-paradoxical iff B is.

Now we're ready for the last step.

Theorem 6. *If $D \subset S^2$ is a countable set, then S^2 and $S^2 \setminus D$ are $SO(3)$-equidecomposable.*

This proof is a little more complicated than the others, so I'll skip it in the talk. But here it is, just in case.

Proof. Let L again be a line through the origin that misses D. Let A be the set of angles θ so that rotation through some multiple of θ takes some point $P \in D$ back into D. Then A is countable. Take $\theta \notin A$, and let ρ be the rotation by θ, so that ρ^n never takes any point of D back into itself. Then $\rho^n(D) \cap \rho^m(D) = \emptyset$ for every m, n.

Now let $\tilde{D} = \bigcup_n \rho^n(D)$. Then $S^2 = \tilde{D} \cup (S^2 \setminus D)$ is equivalent to $\rho(\tilde{D}) \cup (S^2 \setminus D)$, and since the only copy of D in that latter thing gets moved away by ρ, that's $S^2 \setminus D$. □

Corollary (BTP, Weak Form): S^2 is $SO(3)$-paradoxical, as is any sphere centered at the origin. Moreover, any solid ball in $\mathbb{R}^3$ is $E(3)$-paradoxical.

The first two assertions follow immediately from what we've done. For the third, first note that $B \setminus \{(0,0,0)\}$ is $E(3)$-paradoxical, since retracts onto S^2. Then use an argument similar to the one I'm skipping above to deal with the origin.

Corollary (BTP, Strong Form): If A and B are any two bounded subsets of $\mathbb{R}^3$, each with nonempty interior, then they are $E(3)$-equidecomposable.

One final note for the analysts: the pieces are clearly not Lebesgue-measurable. It's an open question whether there is a BTP-style partition using Borel pieces. More precisely, is it true that no compact metric space is paradoxical (w.r.t. isometries) using Borel pieces? Here's another one: can the rigid motions in BTP be chosen so that the pieces never overlap?