Maximal Cohen–Macaulay Modules over the Generic Determinant

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This all stems from a question of G. Bergman.

Setup: Let k be a field, $n \geq 1$, and $X = (x_{ij})$ the generic $n \times n$ matrix over k.

Then it's a standard linear algebra fact (essentially Cramer's rule) that

$$X \cdot \mathrm{adj}(X) = \det(X) \cdot I_n$$

where $\mathrm{adj}(X)$ is the *classical adjoint* of X (the matrix of cofactors, or signed submaximal minors).

Question (Bergman): Can this factorization be refined? That is, is there a further factorization of X and/or $\mathrm{adj}(X)$?

It's fairly easy to see that $\det(X)$ is an irreducible polynomial, so X never factors.

Note that $\det(\mathrm{adj}(X)) = (\det(X))^{n-1}$.

So the real question is:

Can $\mathrm{adj}(X)$ be written as
$$\mathrm{adj}(X) = YZ$$
for noninvertible square matrices Y and Z?

It's a strange question, but the answer is even stranger.

Theorem (Bergman): Assume k is algebraically closed of characteristic zero.

- If n is odd, then $\mathrm{adj}(X)$ cannot be factored.
- If n is even, then any factorization must satisfy either $\det(Y) = \det(X)$ or $\det(Z) = \det(X)$, up to a unit of $k[x_{ij}]$. (I.e., one factor must have "rank one", in a sense to be made clear later.)

The proof is stranger still. From a putative factorization $\mathrm{adj}(X) = YZ$, Bergman constructs a map on Grassmannian varieties which, when $n = 3$, amounts to a nonvanishing tangent vector field on S^2 – a combing of the hairy sphere!

For other odd n, and for the partial result for even n, he uses generalizations of the Hairy Sphere Theorem due to De Concini and Reichstein.

Why do we care?

Recall that an equation of the form

$$AB = fI_n = BA,$$

for some element f in a regular ring S and square matrices A and B over S, is called a *matrix factorization* of f.

Theorem (Eisenbud): Matrix factorizations with no unit entries, up to matrix equivalence, are in 1-1 correspondence with the maximal Cohen–Macaulay modules over the hypeprsurface $S/(f)$, up to isomorphism.

Correspondence: $(A, B) \leftrightarrow \operatorname{cok}(A)$

(Recall that an R-module M is MCM if $\mathrm{depth}(M) = \dim(R)$.)

So, in particular,

$$L := \operatorname{cok} X$$

and

$$M := \operatorname{cok} \operatorname{adj}(X)$$

are MCM modules over the hypersurface $k[X]/\det(X)$.

Since L and M are MCM over the hypersurface, they have projective dimension one over $S = k[x_{ij}]$.

$$0 \longrightarrow F \xrightarrow{\quad X \quad} G \longrightarrow L \longrightarrow 0$$

$$0 \longrightarrow G \xrightarrow{\operatorname{adj}(X)} F \longrightarrow M \longrightarrow 0$$

- Both L and M are indecomposable R-modules.
- Each is the first syzygy of the other (over R)
- $\operatorname{rank}(L) = 1$ and $\operatorname{rank}(M) = n - 1$

When $n = 2$,

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad \text{and} \qquad \operatorname{adj}(X) = \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

So

$$\begin{aligned} M &= \operatorname{cok}(\operatorname{adj}(X)) \\ &\cong \operatorname{cok} X^T && \text{(matrix transpose)} \\ &\cong L^* && \text{(dual into R)} \end{aligned}$$

(since L is its own second syzygy.)

In fact, when $n = 2$ and k is algebraically closed, $k[X]/\det(X)$ has only 3 indecomposable MCM modules up to isomorphism: L, L^*, and R itself. [Buchweitz-Greuel-Schreyer '87]

When $n > 2$, $R = k[X]/\det(X)$ has infinitely many indecomposable MCM modules. (This follows from Auslander's theorem, since R does not have an isolated singularity.)

However, it still has only 3 rank-one MCM modules: L, L^*, and R again [Bruns-Vetter '88].

Other than that, very little is known about MCM modules over R.

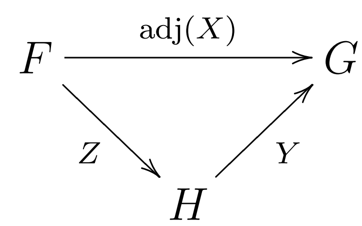
For example,

Question: Is it possible that there are only finitely many MCM R-modules in each rank? Or perhaps a few nicely parametrized families of them?

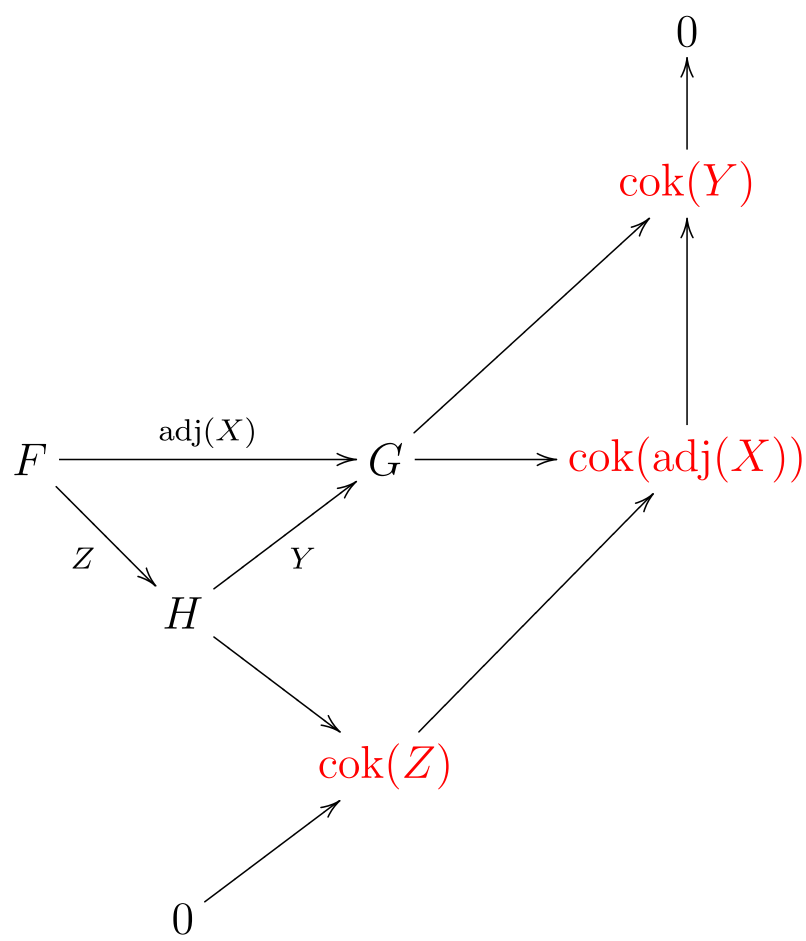
Recall that we're interested in factorizations $\mathrm{adj}(X) = YZ$.

We saw how X and $\mathrm{adj}(X)$ relate to MCM modules. What about factorizations?

A factorization



gives, by the Ker-Coker Lemma,



(The 0s are because $\mathrm{adj}(X)$ is injective.)

So we have an exact sequence

$$0 \longrightarrow \mathrm{cok}(Z) \longrightarrow M \longrightarrow \mathrm{cok}(Y) \longrightarrow 0$$

And Bergman's question becomes:

Does the cokernel of the adjoint matrix
appear as an extension
of two MCM modules?

This is also a slightly strange question.

Translation (to a more answerable question):

Suppose we have a factorization $\operatorname{adj}(X) = YZ$. If we're going to look for extensions

$$0 \longrightarrow \operatorname{cok}(Z) \longrightarrow M \longrightarrow \operatorname{cok}(Y) \longrightarrow 0,$$

then by Bergman's result, we should expect either $\det(Y) = \det(X)$ or $\det(Z)$ = $\det(X)$, up to a unit.

In terms of modules, that means that either $\operatorname{cok}(Y)$ or $\operatorname{cok}(Z)$ has rank 1.

But we know the rank-one MCM modules: L, L^*, and R. Since Y and Z are noninvertible, we rule out R.

Suppose $\operatorname{cok}(Y)$ has rank one, so either $\operatorname{cok}(Y) \cong L$ or $\operatorname{cok}(Y) \cong L^*$. Then form a pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \operatorname{cok}(Y) & \longrightarrow & Q & \longrightarrow & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \operatorname{cok}(Z) & = & \operatorname{cok}(Z) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & o & & & &
\end{array}
$$

The top row is then an extension of L by either L or L^*, that is, an element either of $\operatorname{Ext}^1_R(L, L)$ or $\operatorname{Ext}^1_R(L, L^*)$.

Prop: $\mathrm{Ext}^1_R(L, L) = 0$.

Proof: Since L is a MCM module, $\mathrm{Ext}^1_R(L, L)$ is supported on the singular locus of R, which has dimension 3. So $\mathrm{Ext}^1_R(L, L)$ has codimension at least 3.

Now apply $\mathrm{Hom}_R(L, -)$ to

$$0 \longrightarrow L \longrightarrow R \longrightarrow R/L \longrightarrow 0$$

where L is embedded into R as an ideal linked to a complete intersection. Count depths in the result, using that $\mathrm{Hom}_R(L, R/L)$ is the normal module of R/L, so has codepth 1 over R, to see that $\mathrm{Ext}^1_R(L, L)$ must have codepth at most 2, a contradiction.

What about $\mathrm{Ext}^1_R(L, L^*)$**?**

Main Theorem:

$$6 = \binom{4}{2}.$$

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More to the point,

Main Corollary: $\operatorname{Ext}^1_R(L, L^*)$ is generated by $\binom{n}{2}$ elements.

Much credit to Macaulay 2.

Proof later. But what are those generators?

Main Theorem (linear algebra version):

Let A be an **alternating matrix**, so $A^T = -A$ and $a_{ii} = 0$ for all i. Define another alternating matrix B by

$$b_{ij} = \sum_{k<l} a_{kl}(-1)^{k+l+i+j}[ki \hat{|} lj](X)$$

where $[ki \hat{|} lj](X)$ is the (unsigned) $(n-2) \times (n-2)$ minor of X obtained by dropping the rows k and i and columns l and j.

Then B is the unique solution to

$$\mathrm{adj}(X)^T A = BX.$$

If A is invertible, then

$$\mathrm{adj}(X) = (A^{-1})^T X^T B^T$$

is a nontrivial factorization of $\mathrm{adj}(X)$.

Recall that there exist invertible alternating matrices of size n iff n is even. For example,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

So this answers the remaining case of Bergman's question: the adjoint does indeed factor when n is even.

In fact, it has **many** factorizations.

We also have a two-sided version.

Theorem: Given two alternating matrices A and A', there is a matrix C with entries from $I_1(A) \cdot I_{n-3}(X) \cdot I_1(A') \subseteq R$, and an element $r \in R$ such that

$$\operatorname{adj}(X) = A^{-1}(rX^T + X^T C X^T)(A')^{-1}$$

So a multiple of X^T can be factored out of either side of $\operatorname{adj}(X)$. This corresponds to $\operatorname{cok}(Z)$ having rank one, rather than $\operatorname{cok}(Y)$.

Is that all of them?

Note that a matrix equation of the form

$$\mathrm{adj}(X)^T A = BX$$

gives a commutative diagram

$$\begin{CD}
0 @>>> G @>X>> F @>>> L @>>> 0 \\
@. @VAVV @VVBV @. @. \\
0 @>>> G^* @>\mathrm{adj}(X)^T>> F^* @>>> M^* @>>> 0
\end{CD}$$

and so gives a homomorphism from $L = \operatorname{cok} X$ to $M^* = \operatorname{cok} \mathrm{adj}(X)^T$.

Let's compute $\mathrm{Hom}(L, M^*)$.

Prop: $\mathrm{Hom}_R(L, M^*)$ is a MCM R-module with minimal S-free resolution

$$0 \longrightarrow Alt_n(S) \xrightarrow{B \mapsto X^T B X} Alt_n(S) \longrightarrow \mathrm{Hom}_R(L, M^*) \longrightarrow 0$$

Together with the two-sided factorization theorem and Bruns-Vetter's classification of rank-one modules, this implies that yes, we found them all.

What about the corresponding MCM modules?

Let

$$0 \longrightarrow L^* \longrightarrow Q \longrightarrow L \longrightarrow 0$$

be an element of $\mathrm{Ext}^1_R(L, L^*)$ arising from

$$\mathrm{adj}(X)^T A = BX$$

for some **invertible** alternating matrix A. (So n is necessarily even.)

Then

$$Q \cong \mathrm{cok}(X^T A^{-1} X)$$

Corollary: If Q and Q' are two middle terms of extensions in $\operatorname{Ext}^1_R(L, L^*)$, arising from invertible alternating matrices A and A', then $Q \cong Q'$ if and only if $A = A'$.

That is, the modules coming from answers to Bergman's question are all nonisomorphic.

- $\operatorname{rank} Q = 2$
- $\mu(Q) = n$
- Q is orientable (i.e., $[Q] = 0$ in the class group)

If, on the other hand, A is not invertible (so now n is arbitrary), and

$$\operatorname{adj}(X)^T A = BX$$

then A induces an extension

$$0 \longrightarrow L^* \longrightarrow Q \longrightarrow L \longrightarrow 0$$

where

$$Q \cong \operatorname{cok} \begin{bmatrix} X & A \\ 0 & X^T \end{bmatrix}$$

Again, Q is an orientable module of rank 2, and $n \leq \mu(Q) \leq 2n$.

Prop: The minimal free resolution of $\mathrm{Ext}^1_R(L, L^*)$ over S is of the form

$$
\begin{array}{ccccccccccccc}
 & & & & & & \mathbb{S}_2 G & & & & & & \\
 & & & & & \nearrow & & \searrow & & & & & \\
0 & \longrightarrow & \Lambda^2 F & \longrightarrow & F \otimes G & & & & G \otimes F & \longrightarrow & \Lambda^2 G & \longrightarrow & \operatorname{Ext}^1_R(L, L^*) \longrightarrow 0 \\
 & & & & & \searrow & & \nearrow & & & & & \\
 & & & & & & \mathbb{D}_2 G & & & & & &
\end{array}
$$

$$
\begin{array}{cccccc}
 & & & \binom{n+1}{2} & & \\
0 & \binom{n}{2} & n^2 & & n^2 & \binom{n}{2} \\
 & & & \binom{n+1}{2} & &
\end{array}
$$

Corollary: $\mathrm{Ext}^1_R(L, L^*)$ is a perfect S-module of grade 4, supported on the singular locus of R. On that locus, it represents a MCM module of rank one, isomorphic to the ideal generated by the maximal minors of $n-2$ fixed columns of X in the ring $S/I_{n-1}(X)$..

From this complete description of Ext and the explicit formula for the unique B so that

$$\mathrm{adj}(X)^T A = BX,$$

we see that to an alternating matrix A we can associate the ideal of entries of B, which lives in $I_{n-1}(X)/I_{n-2}(X)$.

If the degrees of the entries of the matrices A are fixed, the set of all rank-two orientable modules arising seems to form a nice variety (it is a subvariety of a Grassmannian over $S/I_{n-2}(X)$).

However, if the entries are left free, there are far far too many to be parametrized by the points of any finite-dimensional variety.

And there may still be other rank-two modules that don't arise in this way.