Finite, Countable, and Bounded CM type

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Notation: $(R, \mathfrak{m}, k)$ is a complete local ring
(graded if time allows at the end)
Usually $k = \mathbb{C}$.
Always Cohen–Macaulay $(\operatorname{depth} R = \dim R)$
An R-module M is *MCM* if $\operatorname{depth} M = \dim R$.

Definition 1. Say R has $\left\{ \begin{matrix} finite \\ countable \\ bounded \end{matrix} \right\}$ Cohen–Macaulay type if there are $\left\{ \begin{matrix} \text{only finitely many, up to } \cong \\ \text{only countably many, up to } \cong \\ \text{a bound on the multiplicities of the} \end{matrix} \right\}$ indecomposable MCM R-modules.

Note: This is a different taxonomy from the finite/discrete/tame/wild system discussed by Drozd.

Examples: simple hypersurfaces, Veronese rings $\mathbb{C}[[x^n, x^{n-1}y, \ldots, y^n]]$ both have FCMT. We'll see others as we go along.

We want to understand the implications between FCMT, CCMT, BCMT, and what each implies about the ring.

Three cases where most is known: hypersurfaces (Gorenstein), isolated singularities, and dimension one.

Theorem 2. *Let $R = \mathbb{C}[[x_1, \ldots, x_d]]/(g)$ be a hypersurface.*

- *(Knörrer '87, Buchweitz-Greuel-Schreyer '87) R has FCMT iff $g = f + x_3^2 + \cdots + x_d^2$ with f one of*

  (A_n) $x_1^2 + x_2^{n+1}$
  (D_n) $x_1^2 x_2 + x_2^{n-1}$
  (E_6) $x_1^3 + x_2^4$
  (E_7) $x_1^3 + x_1 x_2^3$
  (E_8) $x_1^3 + x_2^5$
- *(ditto) R has CCMT iff $g = f + x_3^2 + \cdots + x_d^2$ with f of the form* **ADE** *or*

  (A_∞) x_1^2
  (D_∞) $x_1^2 x_2$
- *(L.-Wiegand '03) R has BCMT iff $g = f + x_3^2 + \cdots + x_d^2$ with f one of the above.*

Corollary 3. *BCMT $\Longleftrightarrow$ CCMT for complete hypersurfaces (over a field of characteristic not 2, 3, 5.*

Questions. Is this true more generally?

Theorem 4. *(Herzog '78) A Gorenstein complete local ring of FCMT or BCMT is a hypersurface as above.*

Questions. What about Gorenstein non-hypersurfaces of CCMT?

Observe: the ones of FCMT are all isolated singularities, while the last two have one-dimensional singular locus.

Theorem 5. *(Auslander '86) A complete CM local ring of FCMT has at most an isolated singularity.*

Proof Sketch. (Huneke-L. '02, does not need "complete")
ETS that $\operatorname{Ext}^1_R(M, N)$ has finite length for all M, N MCM. So take

$$\chi : 0 \to N \to X \to M \to 0$$

and start multiplying by powers of an element r.

$$r^n\chi : 0 \to N \to X_n \to M \to 0$$

Since there are only finitely many choices for X_n, get a repetition $X_a \cong X_b$. Short argument involving Miyata's "almost split" theorem finishes. □

In fact, we have a sort of converse.

Theorem 6. *(Brauer-Thrall theorem, Yoshino '87) Assume R is complete over $\mathbb{C}$. Then R has FCMT if and only if R has BCMT and an isolated singularity.*

Conjecture: (Schreyer '87) CCMT implies one-dimensional singular locus.

Theorem 7. *(Huneke-L. '02) The conjecture holds if R is complete or excellent and k is uncountable.*

Proof Sketch. Let

$$\Lambda = \{p \in \operatorname{Spec} R : \dim R/p = 1 \text{ and } p = \operatorname{Ann}\operatorname{Ext}^1_R(M, N)\}$$

If the singular locus has dimension ≥ 2, then use countable prime avoidance to find $q \notin \Lambda$ with $\dim R/q = 1$. Then check that

$$q = \operatorname{Ann}\operatorname{Ext}^1_R(syz^{d-1}(R/q), syz^d(R/q)).$$

□

Questions. • Is there a Brauer-Thrall theorem for CCMT?

- Does CCMT imply FCMT on the punctured spectrum (know: it localizes)
- (weaker) CCMT + isol.sing $\implies$ FCMT ??
- ascent of CCMT to completion?

Dimension one:

Theorem 8. *(Drozd-Roĭter '67, Green-Reiner '78, Jacobinski '76) Let R be one-dimensional CM local over $\mathbb{C}$. TFAE:*

- *R has FCMT*
- *R has BCMT and $\widehat{R}$ is reduced*
- *$\widehat{R}$ birationally dominates one of A_n, D_n, E_6, E_7, E_8.*

Example: $\mathbb{C}[[t^3, t^4, t^5]]$ birationally dominates E_6 and E_8, so has FCMT.

Theorem 9. *(L-Wiegand '03) A complete $\mathbb{C}$-algebra R of dim. 1 has BCMT but not FCMT iff R is isomorphic to one of*

- $\mathbb{C}[[x, y]]/(y^2)$
- $\mathbb{C}[[x, y]]/(xy^2)$
- $\mathbb{C}[[x, y, z]/I_2 \left(\begin{smallmatrix} x & y & z \\ y & z & z \end{smallmatrix}\right)$

Note: That third ring also has CCMT! Again, we wonder if CCMT is equivalent to BCMT.

Finally: The standard graded rings of FCMT are completely known. In dimension 1, they come from the theorem above; in dim. 2, they are rings of invariants $\mathbb{C}[x, y]^G$ for a cyclic finite group G; there are two in dim. 3.

- $\mathbb{C}[x, y, z, u, v]/I_2 \left(\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right)$
- $\mathbb{C}[x^2, xy, xz, y^2, yz, z^{]}]$

That's it.
They all have minimal multiplicity (in fact, rational singularities).

Questions. ??