

THE F -SIGNATURE AND STRONG F -REGULARITY

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ABSTRACT. We show that the F -signature of a local ring of characteristic p , defined by Huneke and Leuschke, is positive if and only if the ring is strongly F -regular.

In [7], Huneke and Leuschke define the F -signature of an F -finite local ring of prime characteristic with perfect residue field. The F -signature, denoted $s(R)$, is an asymptotic measure of the proportion of R -free direct summands in a direct-sum decomposition of R^{1/p^e} , the ring of p^e th roots of R . This proportion seems to give subtle information on the nature of the singularity defining R . For example, the F -signature of any of the two-dimensional quotient singularities (A_n) , (D_n) , (E_6) , (E_7) , (E_8) is the reciprocal of the order of the group G defining the singularity [7, Example 18]. The main theorem of [7] on F -signatures is as follows.

Theorem 0.1. [7, Theorem 11] *Let (R, \mathfrak{m}, k) be a reduced complete F -finite Cohen–Macaulay local ring containing a field of prime characteristic p . Assume that k is perfect. Then*

- (1) *If $s(R) > 0$, then R is weakly F -regular.*
- (2) *If in addition R is Gorenstein, then $s(R)$ exists, and is positive if and only if R is weakly F -regular.*

(See below for definitions of the F -signature and weak F -regularity.) In this note, we extend this theorem in two directions: we remove the assumption in (2) that R be Gorenstein, and we replace “weakly F -regular” by “strongly F -regular” throughout. Our main theorem is thus as follows.

Theorem 0.2. *Let (R, \mathfrak{m}, k) be a reduced excellent F -finite local ring containing a field of characteristic p , and let $d = \dim R$. Then the following are equivalent:*

- (1) $\liminf \frac{\alpha_q}{q^{d+\alpha(R)}} > 0$.
- (2) $\limsup \frac{\alpha_q}{q^{d+\alpha(R)}} > 0$.
- (3) R is strongly F -regular.

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In particular, if the F -signature $s(R)$ is known to exist, then $s(R)$ is positive if and only if R is strongly F -regular.

We also extend the definition of the F -signature to the case of an imperfect residue field. This allows us to prove that $s(R)$ behaves well with respect to localization (Proposition 1.3).

Our results do not address the existence of the limit defining $s(R)$. Yao has shown that $s(R)$ exists whenever R is Gorenstein on the punctured spectrum [10].

1. THE MAIN RESULT

Throughout what follows, (R, \mathfrak{m}, k) is a reduced Noetherian local ring of dimension d , containing a field of positive characteristic p . We use q to denote a varying power of p . Set $d = \dim(R)$ and $\alpha(R) = \log_p[k : k^p]$. We assume throughout that R is F -finite, that is, the Frobenius endomorphism $F : R \rightarrow R$ defined by $F(r) = r^p$ is a module-finite ring homomorphism. Equivalently, for each $q = p^e$, $R^{1/q} = \{r^{1/q} \mid r \in R\}$ is a finitely generated R -module. In particular, this implies that $\alpha(R) < \infty$, and that R is excellent [8, Propositions 1.1 and 2.5]. Also, when computing length over R , we have $\lambda(R/I^{[q]}R) = \lambda(R^{1/q}/IR^{1/q})/q^{\alpha(R)}$.

We first define the F -signature of R .

Definition 1.1. *Let (R, \mathfrak{m}, k) be as above. For each $q = p^e$, decompose $R^{1/q}$ as a direct sum of finitely generated R -modules $R^{a_q} \oplus M_q$, where M_q has no nonzero free direct summands. The F -signature of R is*

$$s(R) = \lim_{q \rightarrow \infty} \frac{a_q}{q^{d+\alpha(R)}},$$

provided the limit exists.

Our formulation differs slightly from the original definition in [7], where it is assumed that k is perfect, or equivalently that $\alpha(R) = 0$. This reformulation allows us to show that $s(R)$ cannot decrease upon localization. We use a lemma due to Kunz ([8]).

Lemma 1.2. *Let R be an F -finite Noetherian ring of characteristic p . Then for any prime ideals $P \subseteq Q$ of R , $[k(P) : k(P)^p] = [k(Q) : k(Q)^p]p^{\dim R_Q/PR_Q}$. In other words, $\alpha(R_P) = \alpha(R_Q) + \text{ht } Q/P$.*

Proposition 1.3. *Let (R, \mathfrak{m}) be an F -finite local ring and P a prime ideal. For $q = p^e$, let a_q be the number of nonzero R -free direct summands in $R^{1/q}$, and let b_q be the corresponding quantity for R_P . Then*

$$\frac{b_q}{q^{\dim(R_P)+\alpha(R_P)}} \geq \frac{a_q}{q^{\dim(R)+\alpha(R)}}.$$

In particular, if both $s(R)$ and $s(R_P)$ exist, then $s(R_P) \geq s(R)$.

Proof. We have $(R_P)^{1/q} \cong (R^{1/q})_P$, so the number of R_P -free direct summands in $(R_P)^{1/q}$ is at least the number of R -free summands in $R^{1/q}$. A straightforward computation using Lemma 1.2 now gives the result. \square

We now begin to work toward showing that $s(R)$ is positive if and only if R is strongly F -regular. We refer the reader to [6] for basic notions concerning the theory of tight closure, including *finitistic tight closure*, but review briefly the ideas used in the proof.

A Noetherian ring R of characteristic p is said to be *weakly F -regular* provided every ideal of R is tightly closed. Equivalently, the zero module is finitistically tightly closed in $E = E_R(k)$, the injective hull of the residue field of R . In other symbols, $0_E^{*fg} = 0$. We say that R is *strongly F -regular* if for every $c \in R$ not in any minimal prime of R , the inclusion $Rc^{1/q} \subset R^{1/q}$ splits for $q \gg 0$. Equivalently, the zero module is tightly closed in E , that is, $0_E^* = 0$. Weak and strong F -regularity are conjecturally equivalent, but this is known only in low dimension and in some special cases.

A *test element* for R is an element c , not in any minimal prime of R , such that $cI^* \subseteq I$ for every ideal I of R , and the *test ideal*, denoted $\tau(R)$, is the ideal generated by all test elements. For a reduced local ring R , $\tau(R) = \text{Ann}_R 0_E^{*fg}$ by [5, Theorem 8.23]. Thus R is weakly F -regular if and only if $\tau(R) = R$. On the other hand, the *CS test ideal*, cf. [9] and [2], is the ideal $\tilde{\tau}(R) = \text{Ann}_R 0_E^*$. By work of [9] and [2], the CS test ideal behaves well under localization, so defines the non-strongly F -regular locus of $\text{Spec}(R)$. In particular, R is strongly F -regular if and only if $\tilde{\tau}(R) = R$.

It is known that a weakly F -regular ring is *F -pure*, that is, the Frobenius morphism is a pure homomorphism, and that for an F -pure ring both $\tau(R)$ and $\tilde{\tau}(R)$ are radical ideals.

A local ring (R, \mathfrak{m}, k) is said to be *approximately Gorenstein* provided there is a sequence $\{I_t\}$ of \mathfrak{m} -primary irreducible ideals cofinal with the powers of \mathfrak{m} . When R is Cohen–Macaulay and has a canonical ideal J (so is Gorenstein at all associated primes), such a family can be obtained as follows: Let x_1, \dots, x_d be a system of parameters such that $x_1 \in J$ and x_2, \dots, x_d form a system of parameters for R/J . Then $I_t := (x_1^{t-1}J, x_2^t, \dots, x_d^t)R$, for $t \geq 1$, gives the required family. Furthermore, the direct limit $\varinjlim R/I_t$, where the maps in the direct system are $R/I_t \xrightarrow{x_1 \cdots x_d} R/I_{t+1}$, is isomorphic to $E_R(k)$. If $u_1 \in R$ is a representative for the socle generator of R/I_1 , then $u_t := (x_1 \cdots x_d)^{t-1}u_1$ generates the socle of R/I_t , and each u_t maps in the limit to u , the socle element of $E_R(k)$.

More generally ([4, Thm. 1.7]), if R is any locally excellent Noetherian ring that is locally Gorenstein at associated primes, then R is approximately Gorenstein.

The following result of Hochster, together with its corollary below, explains our interest in approximately Gorenstein rings. It can be thought of as a generalization of [7, Lemma 12].

Proposition 1.4. [4, Theorem 2.6] *Let (R, \mathfrak{m}) be an approximately Gorenstein local ring and let $\{I_t\}$ be a sequence of irreducible ideals cofinal with the powers of \mathfrak{m} . Let $f : R \rightarrow M$ be a homomorphism of finitely generated R -modules. Then f is a split injection if and only if $f \otimes_R R/I_t$ is injective for every t .*

Proposition 1.5. *Let (R, \mathfrak{m}) be an approximately Gorenstein local ring with a family of irreducible ideals $\{I_t\}$ as above, and let $u_t \in R$ represent a socle generator for R/I_t . Let $f : R \rightarrow M$ be a homomorphism of finitely generated R -modules. If M has no free summands, then there exists $t_0 > 0$ such that $u_t M \subseteq I_t M$ for all $t \geq t_0$.*

Proof. By Proposition 1.4, $f \otimes R/I_t$ fails to be injective for some t . Since u_t is the unique socle element of R/I_t , we have $f(u_t) \in I_t M$, that is, $u_t M \subseteq I_t M$. This continues to hold for all $t' \geq t$, since there is an injection $R/I_t \rightarrow R/I_{t'}$ with $u_t \mapsto u_{t'}$. \square

We also use a result of Aberbach, which says that, in some sense, elements not in tight closures are very far from being in Frobenius powers.

Theorem 1.6. [1, Prop. 2.4] *Let (R, \mathfrak{m}) be an excellent local domain such that the completion is also a domain. Let $N = \varinjlim R/J_t$ be a direct limit system of cyclic modules. Fix $u \notin 0_N^*$. Then there exists q_0 such that*

$$\bigcup_t (J_t^{[q]} : u_t^q) \subseteq \mathfrak{m}^{[q/q_0]}$$

for all $q \gg 0$ (where the sequence $\{u_t\}$ represents $u \in N$ and $u_t \mapsto u_{t+1}$).

Proof of Theorem 0.2. The Cohen-Macaulayness of R is forced by the assumptions ([7, Theorem 11] and [5]), so we may assume throughout that R is Cohen-Macaulay.

That (1) implies (2) is trivial. So assume that (2) holds. We proceed by induction on the dimension d , the case $d = 0$ being trivial. If $d > 0$, then Proposition 1.3 shows that we may assume by induction on d that R is strongly F -regular on the punctured spectrum. We will show that $0_E^* = 0$, where as above $E = E_R(k)$ is the injective hull of the residue field of R .

Since $\tilde{\tau}(R) = \text{Ann}_R 0_E^*$ is a radical ideal and is known to define the non-strongly F -regular locus of R (see [2]), and R is strongly F -regular on the punctured spectrum, $\text{Ann}_R 0_E^*$ contains the maximal ideal \mathfrak{m} . If $\tilde{\tau}(R) = R$, then we are done, so we assume $\tilde{\tau}(R) = \mathfrak{m}$. Then $0_E^* = \text{soc}(E)$.

As in the discussion above, $E = E_R(k) \cong \varinjlim R/I_t$ for a family of irreducible ideals I_t . Let u be a socle generator for E and $\{u_t\} \subseteq R$ a sequence of representatives for the socle generators of R/I_t , converging to u .

Fix a power q of the characteristic, and decompose $R^{1/q} \cong R^{a_q} \oplus M_q$, where M_q has no nonzero free summands. Then for each t , we have

$$\begin{aligned} \lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t, u_t)^{[q]}\right) &= \frac{\lambda\left(R^{1/q}/I_t R^{1/q}\right)}{q^{\alpha(R)}} - \frac{\lambda\left(R^{1/q}/(I_t, u_t)R^{1/q}\right)}{q^{\alpha(R)}} \\ &= \frac{a_q \lambda(R/I_t) + \lambda(M_q/I_t M_q)}{q^{\alpha(R)}} \\ &\quad - \frac{a_q \lambda(R/(I_t, u_t)) + \lambda(M_q/(I_t, u_t)M_q)}{q^{\alpha(R)}} \\ &= \frac{a_q \lambda(R/I_t) - a_q \lambda(R/(I_t, u_t))}{q^{\alpha(R)}} \\ &\quad + \frac{\lambda(M_q/I_t M_q) - \lambda(M_q/(I_t, u_t)M_q)}{q^{\alpha(R)}} \\ &= \frac{a_q + c_{t,q}}{q^{\alpha(R)}}, \end{aligned}$$

for some $c_{t,q} \geq 0$. By Proposition 1.5, there exists $t_0 > 0$ such that $u_t M_q \subseteq I_t M_q$ for $t \geq t_0$, that is, $c_{t,q} = 0$ for $t \geq t_0$. On the other hand, $\lambda(R/I_t^{[q]}) - \lambda(R/(I_t, u_t)^{[q]}) = \lambda(R/(I_t^{[q]} : u_t^q))$ is equal to 1 for large t since $(I_t^{[q]} : u_t^q) = \mathfrak{m}$ for large t . Thus, for large t ,

$$\lim_{q \rightarrow \infty} \frac{a_q + c_{t,q}}{q^{d+\alpha(R)}} = \lim_{q \rightarrow \infty} \frac{1}{q^{d+\alpha(R)}} = 0,$$

a contradiction.

Lastly, assume that R is strongly F -regular and keep the same notation. We then have $0_E^* = 0$, so $u \notin 0_E^*$. By Theorem 1.6, then, there exists q_0 such that

$$(I_t^{[q]} :_R u_t^q) \subseteq \mathfrak{m}^{[q/q_0]}$$

for all $q \geq q_0$. Fix $q \geq q_0$. Then there exists t_0 such that for all $t \geq t_0$ we have

$$\begin{aligned} \frac{a_q}{q^{\alpha(R)}} &= \lambda\left(R/I_t^{[q]}\right) - \lambda\left(R/(I_t^{[q]}, u_t^q)\right) \\ &= \lambda\left(I_t^{[q]} : u_t^q\right) \\ &\geq \lambda\left(R/\mathfrak{m}^{[q/q_0]}\right). \end{aligned}$$

Divide by q^d and pass to the limit; we see that $\liminf \frac{a_q}{q^{d+\alpha(R)}} \geq e_{HK}(\mathfrak{m}, R)/q_0^d > 0$. Thus (1) holds.

The last statement is immediate if there is a limit. □

The F -signature suggests a form of dimension that we may attach to an F -finite reduced local ring. Let $s_j = \lim_{q \rightarrow \infty} \frac{a_q}{q^{j+\alpha(R)}}$ for $0 \leq j \leq d = \dim(R)$ and set $s_{-1} = 1$. Then we can define the s -dimension of R as $\text{sdim}(R) = \max\{j \geq -1 \mid s_j > 0\}$. A ring which is F -pure then has non-negative s -dimension, and Theorem 0.2 says that R is strongly F -regular if and only if $\text{sdim}(R) = \dim(R)$.

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