ON THE DERIVED CATEGORY OF GRASSMANNIANS IN ARBITRARY CHARACTERISTIC

RAGNAR-OLAF BUCHWEITZ, GRAHAM J. LEUSCHKE, AND MICHEL VAN DEN BERGH

Abstract. In this paper we consider Grassmannians in arbitrary characteristic. Generalizing Kapranov's well-known characteristic-zero results we construct dual exceptional collections on them (which are however not strong) as well as a tilting bundle. We show that this tilting bundle has a quasi-hereditary endomorphism ring and we identify the standard, costandard, projective and simple modules of the latter.

1. Introduction

Throughout K is a field of arbitrary characteristic. Let X be a smooth algebraic variety over K and let $\mathcal{D}$ be its bounded derived category of coherent sheaves. An object $\mathcal{T} \in \mathcal{D}$ is called a *tilting object* if it classically generates $\mathcal{D}$ (i.e. the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{T}$ is $\mathcal{D}$ itself) and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

If $\mathcal{T}$ is a tilting object in $\mathcal{D}$ and $A = \operatorname{End}_{\mathcal{O}_X}(\mathcal{T})$ then the functor $\operatorname{RHom}_{\mathcal{O}_X}(\mathcal{T}, -)$ defines an equivalence $\mathcal{D} \cong D^b(\operatorname{mod} A^\circ)$. If in addition $\mathcal{T}$ is a vector bundle then we call $\mathcal{T}$ a *tilting bundle*.

A sequence of objects $E_1, E_2, \ldots, E_d$ which classically generates $\mathcal{D}$ is called an *exceptional sequence* if $\operatorname{RHom}_{\mathcal{O}_X}(E_j, E_i) = 0$ for $j > i$ and $\operatorname{RHom}_{\mathcal{O}_X}(E_i, E_i) = K$. An exceptional sequence is *strongly exceptional* if in addition $\operatorname{Ext}^k_{\mathcal{O}_X}(E_i, E_j) = 0$ for all i, j and $k > 0$. Obviously if $(E_i)_i$ is strongly exceptional then $\mathcal{T} = \bigoplus_i E_i$ is a tilting object in $\mathcal{D}$.

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Two exceptional sequences $E_1, E_2, \dots, E_d$ and $F_d, F_{d-1}, \dots, F_1$ are said to be *dual* if

$$\operatorname{RHom}_{\mathcal{O}_X}(E_i, F_j) = \delta_{i,j} \cdot K\,.$$

We now specialize to the the case where X is the Grassmannian $\mathbb{G} = \operatorname{Grass}(l, F) \cong \operatorname{Grass}(l, m)$ of l-dimensional subspaces of an m-dimensional K-vector space F. On $\mathbb{G}$ we have a tautological exact sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow F^\vee \otimes_K \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0\,. \tag{1.1}$$

When K is a field of characteristic zero, Kapranov [Kap88] constructs a pair of dual strongly exceptional sequences on $\mathbb{G}$ which we now describe. For a partition α let L^α be the associated Schur functor, let α' be its transpose partition and let $|\alpha| = \sum_i \alpha_i$ be its degree.

Theorem 1.1 (Kapranov [Kap88])**.** *Assume that K has characteristic zero. Let $B_{u,v}$ be the set of partitions with at most u rows and at most v columns equipped with a total ordering $\prec$ such that if $|\alpha| < |\beta|$ then $\alpha \prec \beta$. Let $\overline{B}_{u,v}$ be the same as $B_{u,v}$ but equipped with the opposite ordering. Then there are strongly exceptional sequences on $\mathbb{G}$ given by*

$$(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}} \quad \textit{and} \quad (L^{\alpha'} \mathcal{R})_{\alpha \in \overline{B}_{l,m-l}}\,.$$

In particular the vector bundle

$$\mathcal{K} = \bigoplus_{\alpha \in B_{l,m-l}} L^\alpha \mathcal{Q}$$

is a tilting bundle on $\mathbb{G}$. Moreover the exceptional sequences $(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ and $(L^{\alpha'} \mathcal{R}[|\alpha|])_{\alpha \in \overline{B}_{l,m-l}}$ are dual.

For K a field of positive characteristic p, Kaneda [Kan08] shows that $\mathcal{K}$ remains tilting as long as $p \geqslant m - 1$. However $\mathcal{K}$ fails to be tilting in very small characteristics.

Example 1.2. Assume that K has characteristic 2 and put $\mathbb{G} = \operatorname{Grass}(2, 4)$. Then the short exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 \mathcal{Q} \longrightarrow \mathcal{Q} \otimes \mathcal{Q} \longrightarrow \operatorname{Sym}^2 \mathcal{Q} \longrightarrow 0 \tag{1.2}$$

is non-split. This follows for example from Theorem 5.4 below and the fact that the sequence of GL(2)-representations

$$0 \longrightarrow \textstyle\bigwedge^2 V \longrightarrow V \otimes V \longrightarrow \operatorname{Sym}^2 V \longrightarrow 0$$

is not split, where $V = K^2$ is the standard representation. In particular $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{G}}}(\operatorname{Sym}^2 \mathcal{Q}, \bigwedge^2 \mathcal{Q}) \neq 0$, so that $\operatorname{Sym}^2 \mathcal{Q}$ and $\bigwedge^2 \mathcal{Q}$ are not common direct summands of a tilting bundle on $\mathbb{G}$ in characteristic two.

In this note we give a tilting bundle on $\mathbb{G}$ which exists in arbitrary characteristic. For a partition $\alpha = [\alpha_1, \dots, \alpha_p]$ put

$$\bigwedge^{\alpha} \mathcal{Q} = \bigwedge^{\alpha_1} \mathcal{Q} \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{\alpha_p} \mathcal{Q}\,.$$

Our first main theorem is the following.

Theorem 1.3. *Define a vector bundle on $\mathbb{G}$ by*

$$\mathcal{T} = \bigoplus_{\alpha \in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}\,. \tag{1.3}$$

Then $\mathcal{T}$ is a tilting bundle on $\mathbb{G}$.

In characteristic zero we recover Kapranov's tilting bundle, up to multiplicities, by working out the tensor products in (1.3) using Pieri's formula.

The proof of Theorem 1.3 depends on the following vanishing result which we will also use in [BLVdB13].

Proposition 1.4. *For $\alpha \in B_{l,m-l}$ and β an arbitrary partition we have for $i > 0$*

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^{\beta}\mathcal{Q}) = 0\,. \tag{1.4}$$

Furthermore if $|\beta| < |\alpha|$ then we have as well $\mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^{\beta}\mathcal{Q}) = 0$.

In our next result we show that Kapranov's characteristic-zero result can be partially salvaged in arbitrary characteristic.

Theorem 1.5 (see Theorem 7.4 below)**.** *There exists a total ordering $\prec$ on $B_{l,m-l}$ such that*

$$(L^{\alpha}\mathcal{Q})_{\alpha \in B_{l,m-l}} \quad \textit{and} \quad (L^{\alpha'}\mathcal{R}[|\alpha|])_{\alpha \in \overline{B}_{l,m-l}}$$

are dual exceptional collections on $\mathbb{G}$, where $\overline{B}_{l,m-l}$ is $B_{l,m-l}$ equipped with the opposite ordering.

We use this result to obtain another proof of Kaneda's result that $\mathcal{K}$ remains tilting in characteristics $p \geqslant m-1$ (Corollary 7.6).

The proof of Theorem 1.5 goes through the construction of a nice semi-orthogonal decomposition [BK89] on $D^b(\mathrm{coh}(\mathbb{G}))$ which we summarize in the following theorem.

Theorem 1.6 (see Theorem 5.6 below)**.** *There is a semi-orthogonal decomposition*

$$D^b(\mathrm{coh}(\mathbb{G})) = \left\langle \mathcal{D}_0, \dots, \mathcal{D}_{l(m-l)} \right\rangle$$

where for $d = 0, \dots, l(m-l)$, $\mathcal{D}_d$ is the derived category of the generalized Schur algebra associated to the representations whose composition factors have highest weight $\alpha \in B_{l,m-l}$ such that $|\alpha| = d$.

The connection between Theorem 1.3 and 1.5 depends on the theory of quasi-hereditary algebras [DR92]. In this regard we have the following additional result.

Theorem 1.7. *Let $\mathcal{T}$ be as in Theorem 1.3 and put $A = \operatorname{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T})$. Then A is quasi-hereditary. Furthermore the bundles $(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ correspond to the standard right A-modules and $(L^\alpha \mathcal{R}[|\alpha|])_{\alpha \in \overline{B}_{l,m-l}}$ correspond to the costandard right A-modules.*

This theorem is a special case of Theorem 8.1 below in which we also characterize the simple and projective right A-modules.

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2. Some preliminaries on representation theory

Throughout we use [Jan03] as a convenient reference for facts about algebraic groups. If $H \subset G$ is an inclusion of algebraic groups over the ground field K, then the restriction functor from rational G-modules to rational H-modules has a right adjoint denoted by ind_H^G ([Jan03, I.3.3]). Its right derived functors are denoted by $R^i \operatorname{ind}_H^G$. For an inclusion of groups $K \subset H \subset G$ and M a rational K-representation there is a spectral sequence [Jan03, I.4.5(c)]

$$E_2^{pq}\colon R^p \operatorname{ind}_H^G R^q \operatorname{ind}_K^H M \Longrightarrow R^{p+q} \operatorname{ind}_K^G M\,. \tag{2.1}$$

If G/H is a scheme and V is a finite-dimensional H-representation then $\mathcal{L}_{G/H}(V)$ is by definition the G-equivariant vector bundle on G/H given by the sections of $(G \times V)/H$. The functor $\mathcal{L}_{G/H}(-)$ defines an equivalence between the finite-dimensional H-representations and the G-equivariant vector bundles on G/H. The inverse of this functor is given by taking the fiber in $[H] \in G/H$.

If G/H is a scheme then $R^n \operatorname{ind}_H^G$ may be computed as [Jan03, Prop. I.5.12]

$$R^n \operatorname{ind}_H^G M = H^n(G/H, \mathcal{L}_{G/H}(M))\,. \tag{2.2}$$

We now assume that G is a split reductive group with a given split maximal torus and Borel $T \subset B \subset G$. We let $X(T)$ be the character group of T and we identify the elements of $X(T)$ with the one-dimensional representations of T. The set of roots (the weights of $\operatorname{Lie} G$) is denoted by R. We have $R = R^- \coprod R^+$ where the negative roots R^- represent the roots of $\operatorname{Lie} B$. For $\alpha \in R$ we denote the corresponding coroot in $Y(T) = \operatorname{Hom}(X(T), \mathbb{Z})$ [Jan03, II.1.3] by $\alpha^\vee$. The natural pairing between $X(T)$ and $Y(T)$ is denoted by $\langle -, - \rangle$. A weight

$\lambda \in X(T)$ is dominant if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all positive roots α. The set of dominant weights is denoted by $X(T)_+$. The set $X(T)$ is naturally partially ordered by putting $\lambda \leqslant \mu$ if $\mu - \lambda$ is a sum of positive roots.

The following is the celebrated Kempf vanishing result ([Kem76], see also [Jan03, II.4.5]).

Theorem 2.1. *If $\lambda \in X(T)_+$ then $R^i \operatorname{ind}_B^G \lambda = H^i(G/B, \mathcal{L}_{G/B}(\lambda))$ vanishes for all strictly positive i.* □

We now restrict to $G = \mathrm{GL}(m)$. In this case we let T be the diagonal matrices in G and B the lower triangular matrices. The weights of T can be identified with m-tuples of integers $[\alpha_1, \dots, \alpha_m]$ via $\operatorname{diag}(z_1, \dots, z_m) \mapsto z_1^{\alpha_1} \cdots z_m^{\alpha_m}$. Thus $X(T) \cong Y(T) \cong \mathbb{Z}^m$. Under this identification roots and coroots coincide and are given by $(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$. The pairing between $X(T)$ and $Y(T)$ is the standard Euclidean scalar product and hence $X(T)_+ = \{[\alpha_1, \dots, \alpha_m] \mid \alpha_i \geqslant \alpha_j \text{ for } i \leqslant j\}$. A dominant weight with only non-negative entries will be called a *partition*. Mentally we represent a partition by its Young diagram, with the length of the rows corresponding to the entries. The sum $\sum_i \alpha_i$ is the *degree* of the weight α and is denoted by $|\alpha|$. We say that a representation has degree d if all its weights have degree d. We say that a representation is *polynomial* is all its weights contain only non-negative entries.

If $\alpha = [\alpha_1, \dots, \alpha_m]$ is a partition then we denote by L^α, K^α the corresponding *Schur* and *Weyl functors*. More precisely for a vector space (or a vector bundle) V define for a partition α

$$\bigwedge^\alpha V = \bigotimes_i \bigwedge^{\alpha_i} V \qquad \operatorname{Sym}^\alpha V = \bigotimes_i \operatorname{Sym}^{\alpha_i} V \qquad D^\alpha V = \bigotimes_i D^{\alpha_i} V$$

where in particular $D^u V = (V^{\otimes u})^{S_u}$ is the u^{th} divided power representation.

Then we put with $d = |\alpha|$:

$$L^\alpha V = \operatorname{im}\left(\bigwedge^{\alpha'} V \xrightarrow{a} V^{\otimes d} \xrightarrow{s} \operatorname{Sym}^\alpha V\right) \tag{2.3}$$

$$K^\alpha V = \operatorname{im}\left(D^\alpha V \xrightarrow{s} V^{\otimes d} \xrightarrow{a} \bigwedge^{\alpha'} V\right) \tag{2.4}$$

where a and s are respectively the anti-symmetrization map and the symmetrization map. Their precise form is derived from a labeling of the Young diagram associated to α. The resulting representations $K^\alpha V$, $L^\alpha V$ are independent of this labeling.

In the sequel we freely pass between the functor point of view and the representation theory point of view using the following lemma. If $\lambda \in X(T)_+$ then $H^0(\lambda) \overset{\text{def}}{=} \operatorname{ind}_B^G \lambda$ is a so-called *induced representation* with highest weight λ. Dually one defines the corresponding *Weyl representation* as $V(\lambda) = H^0(-w_0\lambda)^\vee$ where w_0 is the longest element of the Weyl group [Jan03, §2.13].

Lemma 2.2. *Let V be the standard representation of $\mathrm{GL}(m)$ and let α be a partition. Then*

$$L^\alpha V = H^0(\alpha) \tag{2.5}$$

$$K^\alpha V = V(\alpha) \tag{2.6}$$

Proof. The identity (2.5) is [Wey03, (4.1.10)][1]. To prove (2.6) we note that by [Jan03, II.2.13(2)] we have $V(\alpha) = {}^\tau H^0(\alpha)$, where ${}^\tau M$ is $M^\vee$ as a vector space and $g \in G$ acts on $\varphi \in M^\vee$ via $g \cdot \varphi = \varphi \circ g^t$ where $(-)^t$ denotes transposition. Clearly $M \mapsto {}^\tau M$ is a contravariant monoidal functor and furthermore one verifies

$$\begin{aligned} {}^\tau \operatorname{Sym}^u V &= D^u V \\ {}^\tau {\textstyle\bigwedge}^u V &= {\textstyle\bigwedge}^u V \,. \end{aligned}$$

Applying ${}^\tau(-)$ to the right-hand side of (2.3) yields the right-hand side of (2.4), finishing the proof. ◻

According to [Jan03, Prop. II.2.4] $L^\alpha V$ has a simple socle which we denote by Σ^α. According to [Jan03, §II.2.6] $K^\alpha V$ has a simple top, which is also equal to Σ^α.

3. Proof of Theorem 1.3 and Proposition 1.4

We stick to the notation already introduced in the introduction. We will identify $\mathbb{G} = \mathrm{Grass}(l, F)$ with $\mathrm{Grass}(m-l, F^\vee)$ via the correspondence $(V \subset F) \mapsto ((F/V)^\vee \subset F^\vee)$.

For convenience we choose a basis $(f_i)_{i=1,\ldots,m}$ for F and a corresponding dual basis $(f_i^*)_i$ for $F^\vee$. We view $\mathbb{G}$ as the homogeneous space G/P with $G = \mathrm{GL}(F^\vee) = \mathrm{GL}(m)$ and $P \subset G$ the parabolic subgroup stabilizing the point $(W \subset F^\vee) \in \mathbb{G}$ where $W = \sum_{i=l+1}^m K f_i^*$. As above let T and B be respectively the diagonal matrices and the lower triangular matrices in G.

Let $H = G_1 \times G_2 = \mathrm{GL}(l) \times \mathrm{GL}(m-l) \subset \mathrm{GL}(m)$ be the Levi-subgroup of P containing T. We put $B_i = B \cap G_i$ and $T_i = T \cap G_i$. We denote the standard representations of G_1 and G_2 by V and W respectively. Thus for $x = [P] \in G/P$ we have $V = \mathcal{Q}_x$ and $W = \mathcal{R}_x$; equivalently $\mathcal{Q} = \mathcal{L}_{\mathbb{G}}(V)$ and $\mathcal{R} = \mathcal{L}_{\mathbb{G}}(W)$. (Throughout we silently view G_i-representations as P-representations to apply $\mathcal{L}_{\mathbb{G}}(-)$.) It follows that $\bigwedge^{\alpha'} \mathcal{Q} = \mathcal{L}_{\mathbb{G}}(\bigwedge^{\alpha'} V)$ and $L^\alpha \mathcal{Q} = \mathcal{L}_{\mathbb{G}}(L^\alpha V)$.

[1] Note that our L^α is $L_{\alpha'}$ in [Wey03].

For use in the proof below we fix an additional parabolic P° in G given by the stabilizer of the flag $(\sum_{i\geqslant p} K f_i^*)_{p=1,\dots,l}$. We let $G^\circ = \mathrm{GL}(m-l+1) \subset P^\circ \subset G = \mathrm{GL}(m)$ be the lower right $(m-l+1 \times m-l+1)$-block in $\mathrm{GL}(m)$. We put $T^\circ = T \cap G^\circ$ and $B^\circ = B \cap G^\circ$, i.e. B° is the set of lower triangular matrices in G° and T° is the set of diagonal matrices.

Proof of Proposition 1.4. It follows from the usual spectral sequence argument that $\mathrm{Ext}^i_{\mathcal{O}_\mathbb{G}}(\bigwedge^{\alpha'}\mathcal{Q}, L^\beta\mathcal{Q})$ is the i^{th} cohomology of $\mathcal{H}om_{\mathcal{O}_\mathbb{G}}(\bigwedge^{\alpha'}\mathcal{Q}, L^\beta\mathcal{Q}) \cong (\bigwedge^{\alpha'}\mathcal{Q})^\vee \otimes_\mathbb{G} L^\beta\mathcal{Q}$, so we must show

$$H^i(\mathbb{G}, \textstyle\bigwedge^{u_1}\mathcal{Q}^\vee \otimes_\mathbb{G} \cdots \otimes_\mathbb{G} \bigwedge^{u_{m-l}}\mathcal{Q}^\vee \otimes_\mathbb{G} L^\beta\mathcal{Q}) = 0$$

for all $i > 0$ and $u_1 \geqslant \cdots \geqslant u_{m-l} \geqslant 0$, and also for $i = 0$ if $\sum u_i < |\beta|$.

Using the identity

$$(\textstyle\bigwedge^u\mathcal{Q})^\vee = \bigwedge^{l-u}\mathcal{Q} \otimes \left(\bigwedge^l\mathcal{Q}\right)^\vee$$

and the characteristic free version of the Littlewood-Richardson rule (see [Bof88] or [Wey03, (2.3.4)]) we reduce immediately to the case $u_1 = \cdots = u_{m-l} = l$. The tautological exact sequence (1.1) allows us to write

$$\left(\textstyle\bigwedge^l\mathcal{Q}\right)^\vee = \bigwedge^m F \otimes_K \bigwedge^{m-l}\mathcal{R}\,.$$

Thus we need to prove that

$$L^\gamma\mathcal{Q} \otimes_\mathbb{G} \textstyle\bigwedge^{m-l}\mathcal{R} \otimes_\mathbb{G} \cdots \otimes_\mathbb{G} \bigwedge^{m-l}\mathcal{R}$$

(with $m-l$ factors of $\bigwedge^{m-l}\mathcal{R}$) has vanishing higher cohomology. Using (2.2) we see that we must prove that for $i > 0$ we have

$$R^i \operatorname{ind}_P^G \left(L^\gamma V \otimes \textstyle\bigwedge^{m-l}W \otimes \cdots \otimes \bigwedge^{m-l}W\right) = 0\,, \tag{3.1}$$

where as above V, W are the standard representations of G_1, G_2. Since V has rank l, we may assume that γ has at most l entries. Put $\overline{\gamma} = [\gamma_1, \dots, \gamma_l, m-l, \dots, m-l] \in X(T)$. We have

$$L^\gamma V \otimes \textstyle\bigwedge^{m-l}W \otimes \cdots \otimes \bigwedge^{m-l}W = \operatorname{ind}_B^P \overline{\gamma}\,.$$

It is clear that $\overline{\gamma}$ is dominant when viewed as a weight for T considered as a maximal torus in $H = G_1 \times G_2$ with respect to the Borel subgroup $B_1 \times B_2$. So Kempf vanishing implies that $R^i \operatorname{ind}_B^P \overline{\gamma} = R^i \operatorname{ind}_{B_1\times B_2}^{G_1\times G_2} \overline{\gamma} = 0$ for all $i > 0$.

Thus the spectral sequence (2.1) degenerates and we obtain

$$R^i \operatorname{ind}_P^G \left(L^\gamma V \otimes \textstyle\bigwedge^{m-l}W \otimes \cdots \otimes \bigwedge^{m-l}W\right) = R^i \operatorname{ind}_B^G \overline{\gamma}\,. \tag{3.2}$$

Thus if $\overline{\gamma}$ is dominant (i.e. $\gamma_l \geqslant m-l$) then the desired vanishing (3.1) follows by invoking Kempf vanishing again.

Assume then that $\overline{\gamma}$ is not dominant, i.e. $0 \leqslant \gamma_l < m-l$. We claim that $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = 0$ for all i. Then by the spectral sequence (2.1) applied to $B \subset P^\circ \subset G$ we obtain that $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i.

To prove the claim we note that $P^\circ/B \cong G^\circ/B^\circ$ and hence by (2.2) $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = R^i \operatorname{ind}_{B^\circ}^{G^\circ}(\overline{\gamma} \mid T^\circ)$. In other words we have reduced ourselves to the case $l=1$ (replacing m by $m-l+1$).

So now we assume $l=1$. Thus $\mathbb{G} = \mathbb{P}(F) \cong \mathbb{P}^{m-1}$, which we write as $\mathbb{P}$ for short. The partition γ consists of a single entry γ_1 and we have $\overline{\gamma} = [\gamma_1, m-1, \dots, m-1]$. Under the assumption $\gamma_1 < m-1$ we have to prove $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i. Applying (3.2) in reverse this means we have to prove that

$$\mathcal{Q}^{\otimes \gamma_1} \otimes_{\mathbb{P}} \left(\bigwedge\nolimits^{m-1} \mathcal{R}\right)^{\otimes m-1}$$

has vanishing cohomology on $\mathbb{P}$.

We now observe $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}}(1)$ and since

$$\mathcal{R} \cong \ker(\mathcal{O}_{\mathbb{P}}^m \longrightarrow \mathcal{O}_{\mathbb{P}}(1))$$

we also have

$$\bigwedge\nolimits^{m-1} \mathcal{R} \cong \mathcal{O}_{\mathbb{P}}(-1)$$

so that

$$\mathcal{Q}^{\otimes \gamma_1} \otimes_{\mathbb{P}} \bigwedge\nolimits^{m-1} \mathcal{R}^{\otimes m-1} \cong \mathcal{O}_{\mathbb{P}}(-m+1+\gamma_1)\,.$$

It is standard that this line bundle has vanishing cohomology when $0 \leqslant \gamma_1 < m-1$, so we are done.

For the last statement of the Proposition, observe that in the above argument if $|\beta| < |\alpha|$ then we are always in the case where $\overline{\gamma}$ is not dominant, and thus the vanishing holds also when $i=0$. $\square$

Proof of Theorem 1.3. The main thing to prove is that $\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$. Applying the characteristic-free Littlewood-Richardson rule [Bof88], we see that it suffices to prove that $\mathcal{T}^\vee \otimes_{\mathbb{G}} L^\gamma \mathcal{Q}$ has vanishing higher cohomology whenever γ is a partition with at most l rows. This is the content of Proposition 1.4.

Kapranov's resolution of the diagonal argument together with the characteristic-free version of Cauchy's formula [Wey03, (2.3.2)] still implies that the vector bundle $\mathcal{K}$ in Theorem 1.1 classically generates $D^b(\operatorname{coh}(\mathbb{G}))$. See, for example, [LSW89]. Thus it suffices to show that $L^\alpha \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ is in the thick subcategory $\mathcal{C}$ generated by $\mathcal{T}$. Assume this is not the case and let α be minimal for the lexicographic ordering on partitions such that $L^\alpha \mathcal{Q}$ is *not* in $\mathcal{C}$.

Consider $\mathcal{U} = \bigwedge^{\alpha'_1}\mathcal{Q}\otimes_{\mathbb{G}}\cdots\otimes_{\mathbb{G}}\bigwedge^{\alpha'_l}\mathcal{Q}$. Then Pieri's formula, which is a special case of the Littlewood-Richardson rule, yields a filtration of $\mathcal{U}$ with successive quotients $L^\beta\mathcal{Q}$ such that $\beta \leqslant \alpha$ and such that $L^\alpha\mathcal{Q}$ appears with multiplicity one. Furthermore $\mathcal{U}$ has a *good filtration* [Jan03, §II.4.16], one in which the $L^\beta\mathcal{Q}$ appearing as quotients are in decreasing order for the lexicographic ordering on partitions, that is, the largest β appear on top [Jan03, II.4.16, Remark (4)]. Hence $\mathcal{U}$ maps surjectively to $L^\alpha\mathcal{Q}$ and the kernel is an extension of various $L^\beta\mathcal{Q}$ with β strictly smaller than α in the lexicographic ordering. By the hypotheses all such $L^\beta\mathcal{Q}$ are in $\mathcal{C}$. Since $\mathcal{U}$ is in $\mathcal{C}$ as well we obtain that $L^\alpha\mathcal{Q}$ is in $\mathcal{C}$, which is a contradiction. ☐

Remark 3.1. By [Don93, Lemma (3.4)] we can obtain the following more economical tilting bundle for $\mathbb{G}$

$$\mathcal{T}^\circ = \bigoplus_{\alpha\in B_{l,m-l}} \mathcal{L}_{\mathbb{G}}(M^\alpha)\,,$$

where M^α is the tilting $\mathrm{GL}(l)$-representation with highest weight α. Note however that the character of M^α strongly depends on the characteristic. Hence so does the nature of $\mathcal{T}^\circ$.

For use below we need the following complement to Proposition 1.4.

Proposition 3.2. *For every partition α and every polynomial G_1-representation U of degree $<|\alpha|$ we have*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\textstyle\bigwedge^{\alpha'}\mathcal{Q},\mathcal{L}_{\mathbb{G}}(U)) = 0\,. \tag{3.3}$$

Proof. It suffices to prove the claimed vanishing for U simple of degree less than $|\alpha|$, so for $U=\Sigma^\beta$ with β a partition such that $|\beta|<|\alpha|$. We do this by induction on β. Since Σ^β is the socle of $L^\beta\mathcal{Q}$ we have by [Jan03, Prop. 6.15] a short exact sequence

$$0\longrightarrow \Sigma^\beta\longrightarrow L^\beta V\longrightarrow S\longrightarrow 0$$

where S is obtained through extensions involving only Σ^γ with $\gamma<\beta$. By induction we may assume $\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'}\mathcal{Q},\mathcal{L}_{\mathbb{G}}(S))=0$. Then (3.3) for $U=\Sigma^\beta$ follows from the final statement of Proposition 1.4. ☐

4. Reminder on semi-orthogonal decompositions

We recapitulate some facts concerning semi-orthogonal decompositions that we need later. No originality is intended.

If $\mathcal{S}$ is a triangulated category and S is a set of objects then we denote by $\langle S\rangle$ the smallest triangulated subcategory of $\mathcal{S}$ that contains S

and is closed under isomorphisms. If $\mathcal{S} = \langle S \rangle$ then we say that S generates $\mathcal{S}$ as a triangulated category. (This is stronger than "classically" generating $\mathcal{S}$ as in the Introduction.)

Definition 4.1. A *semi-orthogonal decomposition* of a triangulated category $\mathcal{S}$ is a sequence of full subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{S}$ generating $\mathcal{S}$ as a triangulated category and such that $\operatorname{Hom}_{\mathcal{S}}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $j > i$. We denote such a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. Sometimes we write $\mathcal{S} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$.

If X is an object in a triangulated category then a filtration F of length n on X is a sequence of maps

$$0 = F_n X \longrightarrow F_{n-1} X \longrightarrow \cdots \longrightarrow F_0 X = X\,.$$

We write $(\operatorname{gr}_F X)_i = \mathsf{cone}(F_{i-1}X \longrightarrow F_i X)$. The following well-known lemma shows that Definition 4.1 is equivalent to the seemingly stronger one in [Kuz09, Def. 2.3].

Lemma 4.2. *Let $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semi-orthogonal decomposition of $\mathcal{S}$. Then every object X in $\mathcal{S}$ has a filtration F of length n such that $(\operatorname{gr}_F X)_i \in \mathcal{A}_{i+1}$.*

Proof. By induction it is sufficient to prove this for $n = 2$. In that case the result is [Bon89, Lemma 3.1]. □

In order to work conveniently with semi-orthogonal decompositions one needs a property called "*admissibility*" [BK89]. If $\mathcal{A}$ is a saturated full triangulated subcategory of a triangulated category $\mathcal{S}$ then $\mathcal{A}$ is (left, right) admissible if the inclusion functor $\mathcal{A} \longrightarrow \mathcal{S}$ has a (left, right) adjoint, or equivalently if there exist semi-orthogonal decompositions $\langle \mathcal{A}, \mathcal{A}' \rangle$ resp. $\langle \mathcal{A}'', \mathcal{A} \rangle$. If $\mathcal{A}$ is both left and right admissible then it is said to be admissible.

A *saturated* triangulated category is a K-linear triangulated category $\mathcal{A}$ such that for all $A,\ B \in \mathcal{A}$ we have $\sum_i \dim \operatorname{Hom}^i_{\mathcal{A}}(A, B) < \infty$ and such that every co- or contravariant cohomological functor $H^i \colon \mathcal{A} \longrightarrow \operatorname{mod}(K)$ satisfying $\sum_i \dim H^i(A) < \infty$ is representable. The derived category of coherent sheaves on a smooth proper algebraic variety is a particular example of a saturated triangulated category [BVdB03, BK89].

If $\mathcal{A}$ is a full triangulated subcategory of a K-linear triangulated category $\mathcal{S}$ then $\mathcal{A}$ is admissible [BK89, Prop 2.6]. Furthermore if $\mathcal{S}$ is a saturated triangulated category then every left/right admissible subcategory is automatically admissible (and hence saturated). This follows by combining [BK89, Prop. 2.6] and [BK89, Prop. 2.8]. From this we deduce that if we have a semi-orthogonal decomposition $\mathcal{S} =$

$\langle\mathcal{A}_1,\dots,\mathcal{A}_n\rangle$ of a saturated $\mathcal{S}$ then all the "slices" $\langle\mathcal{A}_i,\dots,\mathcal{A}_j\rangle$ are admissible and saturated.

In particular if we put $\mathcal{S}_{\leqslant i}=\langle\mathcal{A}_1,\dots,\mathcal{A}_i\rangle$ then this yields a filtration $\mathcal{S}_{\leqslant 1}\subset\cdots\subset\mathcal{S}_{\leqslant n}=\mathcal{S}$ by admissible subcategories. Let $\mathcal{B}_i$ be the right orthogonal of $\mathcal{S}_{\leqslant i-1}$ in $\mathcal{S}_{\leqslant i}$. Then we have semi-orthogonal decompositions $\mathcal{S}_{\leqslant i}=\langle\mathcal{B}_i,\mathcal{S}_{\leqslant i-1}\rangle$. Iterating we obtain a semi-orthogonal decomposition

$$\mathcal{S}=\langle\mathcal{B}_n,\mathcal{B}_{n-1},\dots,\mathcal{B}_1\rangle$$

such that

$$\langle\mathcal{A}_1,\dots,\mathcal{A}_i\rangle=\langle\mathcal{B}_i,\dots,\mathcal{B}_1\rangle\ .$$

This is called the semi-orthogonal decomposition (right) dual to $\mathcal{S}=\langle\mathcal{A}_1,\dots,\mathcal{A}_n\rangle$. Note that [Kuz09, (4)]

$$\begin{aligned}\mathcal{B}_i&=\mathcal{S}_{\leqslant i}\cap\mathcal{S}_{\leqslant i-1}^{\perp}\\&=\langle\mathcal{A}_{i+1},\dots,\mathcal{A}_n\rangle^{\perp}\cap\langle\mathcal{A}_1,\dots,\mathcal{A}_{i-1}\rangle^{\perp}\\&=\langle\mathcal{A}_1,\dots,\mathcal{A}_{i-1},\mathcal{A}_{i+1},\dots,\mathcal{A}_n\rangle^{\perp}\ .\end{aligned}$$

In particular for $i\neq j$

$$\operatorname{Hom}(\mathcal{A}_i,\mathcal{B}_j)=0\,. \tag{4.1}$$

The following is also well-known [Kuz09].

Lemma 4.3. *Let γ_i be the composition of the canonical maps*

$$\gamma_i\colon\mathcal{A}_i\longrightarrow\mathcal{S}_{\leqslant i}\longrightarrow\mathcal{S}_{\leqslant i}\big/\mathcal{S}_{\leqslant i-1}=\mathcal{B}_i\,.$$

Then γ_i is an equivalence of categories. Furthermore we have for $A\in\mathcal{A}_i$, $B\in\mathcal{B}_i$

$$\operatorname{Hom}_{\mathcal{S}}(A,B)=\operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A),B)=\operatorname{Hom}_{\mathcal{S}}(\gamma_i(A),B)\,.$$

Proof. We have semi-orthogonal decompositions

$$\mathcal{S}_{\leqslant i}=\langle\mathcal{S}_{\leqslant i-1},\mathcal{A}_i\rangle=\langle\mathcal{B}_i,\mathcal{S}_{\leqslant i-1}\rangle\ .$$

The fact that γ_i is an equivalence follows from [BK89, Lemma 1.9].

Let $\jmath\colon\mathcal{A}_i\longrightarrow\mathcal{S}_{\leqslant i}$ and $\imath\colon\mathcal{B}_i\longrightarrow\mathcal{S}_{\leqslant i}$ be the inclusion functors and let $\imath^*$ be the left adjoint to $\imath$. Then $\gamma_i=\imath^*\circ\jmath$. We have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{S}}(A,B)&=\operatorname{Hom}_{\mathcal{S}_{\leqslant i}}(\jmath A,\imath B)\\&=\operatorname{Hom}_{\mathcal{S}_{\leqslant i}}(\imath^*\jmath A,B)\\&=\operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A),B)\,.\end{aligned}$$

The equality $\operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A),B)=\operatorname{Hom}_{\mathcal{S}}(\gamma_i(A),B)$ is just that $\mathcal{B}_i$ is a full subcategory of $\mathcal{S}$. $\square$

5. Semi-orthogonal decompositions for Grassmannians

In this section we write $\mathcal{D}$ for the bounded derived category of coherent sheaves on $\mathbb{G}$. This is in particular a saturated category (see §4). We will construct a semi-orthogonal decomposition of $\mathcal{D}$.

We start by observing that the proof of Theorem 1.3 actually shows

Lemma 5.1. *$\mathcal{D}$ is generated as a triangulated category by $(\bigwedge^{\alpha'}\mathcal{Q})_{\alpha\in B_{l,m-l}}$ (instead of just classically generated, see §4).* ☐

A set S of weights is *saturated* if whenever $\alpha \in S$ and $\beta < \alpha$ we have $\beta \in S$. (Here and below "$<$" is the standard ordering on weights; see §2.) The set $B_{l,m-l}$ is an example of a saturated set. For $d \geqslant 0$ let $\mathcal{C}_d$ be the category of finite-dimensional $G_1 = \mathrm{GL}(l)$-representations whose composition factors have highest weights α satisfying $|\alpha| = d$ and $\alpha \in B_{l,m-l}$. Thus $\mathcal{C}_d$ is a *truncated category* in the sense of [Jan03, Ch. A] associated to a saturated set of dominant weights. In particular $\mathcal{C}_d$ is the category of modules over a certain finite-dimensional algebra, called a *generalized Schur algebra* [Jan03, §A.16].

We collect some elementary facts about the derived category of $\mathcal{C}_d$.

Lemma 5.2. *Let $\mathrm{Rep}(G_1)$ be the category of rational G_1-representations and for each d let $D^b_{\mathcal{C}_d}(\mathrm{Rep}(G_1))$ be the bounded derived category of complexes of representations having cohomology in $\mathcal{C}_d$. The canonical functor*

$$D^b(\mathcal{C}_d) \longrightarrow D^b_{\mathcal{C}_d}(\mathrm{Rep}(G_1))$$

is an equivalence of categories.

Proof. This follows from the fact that the Yoneda Ext's in $\mathcal{C}_d$ are the same as those in the ambient category $\mathrm{Rep}(G_1)$ (see [Jan03, Prop. A.10]). ☐

In the sequel we will simply confuse $D^b(\mathcal{C}_d)$ and $D^b_{\mathcal{C}_d}(\mathrm{Rep}(G_1))$.

Lemma 5.3. *The triangulated category $D^b(\mathcal{C}_d)$ is generated by the representations $\bigwedge^{\alpha'}V$ for $\alpha \in B_{l,m-l}$, $|\alpha| = d$, where as usual V is the standard representation of G_1.*

Proof. This is of course well-known but for the convenience of the reader we give the proof. Let $\mathcal{A}$ be the full subcategory of $D^b(\mathcal{C}_d)$ generated by $(\bigwedge^{\alpha'}V)_{\alpha\in B_{l,m-l},|\alpha|=d}$ It is sufficient to prove that $\mathcal{A}$ contains the simples Σ^α for $\alpha \in B_{l,m-l}$, $|\alpha| = d$.

By reasoning similar to the proof of Theorem 1.3 we see that $\mathcal{A}$ contains $K^\alpha V$ for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$. By [Jan03, II.2.13] $K^\alpha V$ has simple top Σ^α and by the dual version of [Jan03, II.6.13] the other

Jordan-Hölder quotients of $K^\alpha V$ are of the form Σ^γ with $|\gamma| = |\alpha| = d$ and $\gamma < \alpha$. Thus $\Sigma^\gamma \in \mathcal{C}_d$. By induction we may assume that such $\Sigma^\gamma \in \mathcal{A}$. Hence it follows that $\Sigma^\alpha \in \mathcal{A}$. $\square$

We define a functor

$$\Phi_d \colon D^b(\mathcal{C}_d) \longrightarrow \mathcal{D}$$

by $\Phi_d(U) = \mathcal{L}_{\mathbb{G}}(U)$ for $U \in \mathcal{D}^b(\mathcal{C}_d)$, where we view U as a complex of P-representations in the obvious way.

Theorem 5.4. *The functor Φ_d is fully faithful.*

Proof. By Lemma 5.3 it is sufficient to prove that for $\alpha, \beta \in B_{l,m-l}$ with $|\alpha| = |\beta| = d$ the canonical map

$$\mathrm{RHom}_{G_1}(\textstyle\bigwedge^{\alpha'} V, \bigwedge^{\beta'} V) \longrightarrow \mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, \bigwedge^{\beta'} \mathcal{Q}) \tag{5.1}$$

is an isomorphism (where we have used that $\mathcal{L}_{\mathbb{G}}(V) = \mathcal{Q}$). Now $\bigwedge^{\alpha'} V$ and $\bigwedge^{\beta'} V$ are tilting representations [Don93, Lemma (3.4)] and so on the left-hand side of (5.1) there are no higher Ext's. Likewise on the right-hand side there are no higher Ext's because of Proposition 1.4.

So we only have to show that the map

$$\mathrm{Hom}_{G_1}(\textstyle\bigwedge^{\alpha'} V, \bigwedge^{\beta'} V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, \bigwedge^{\beta'} \mathcal{Q})$$

is an isomorphism. By the vanishing of higher Ext's and the characteristic-independence of the Euler characteristic it is sufficient to prove this in characteristic zero.

Thus we assume that K has characteristic zero. Then we may decompose $\bigwedge^{\alpha'} V$, $\bigwedge^{\beta'} V$ as sums of simple modules $L^\gamma V$, $L^\delta V$. Thus it is sufficient to prove that

$$\mathrm{Hom}_{\mathrm{GL}(l)}(L^\alpha V, L^\beta V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^\beta \mathcal{Q})$$

is an isomorphism. Or in other words, since in characteristic zero the $L^\alpha V$ are simple,

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^\beta \mathcal{Q}) = \delta_{\alpha,\beta} \cdot K\,.$$

This follows from the Littlewood-Richardson rule, combined with Bott's theorem (see e.g. [Kap88, §3.2, §3.3]). $\square$

Now let $\mathcal{D}_d$ be the essential image of $D^b(\mathcal{C}_d)$ under Φ_d. From Lemma 5.3 we obtain:

Corollary 5.5. *$\mathcal{D}_d$ is generated by $\bigwedge^{\alpha'} \mathcal{Q}$ for $\alpha \in B_{l,m-l}$, $|\alpha| = d$.* $\square$

We have:

Theorem 5.6. *The triangulated category $\mathcal{D}$ has a semi-orthogonal decomposition*

$$\mathcal{D} = \left\langle \mathcal{D}_0, \ldots, \mathcal{D}_{l(m-l)} \right\rangle . \tag{5.2}$$

Furthermore $\mathcal{D}_d$ is generated by $L^\alpha \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$.

Proof. By Lemma 5.1, $\mathcal{D}$ is generated by $\bigwedge^{\alpha'} \mathcal{Q}$ for $\alpha \in B_{l,m-l}$. It follows from Corollary 5.5 that $\mathcal{D}$ is generated by $(\mathcal{D}_d)_d$ and that $\mathcal{D}_d$ is generated by those $\bigwedge^{\alpha'} \mathcal{Q}$ with $|\alpha| = d$.

To complete the proof that (5.2) is a semi-orthogonal decomposition we need that $\operatorname{Hom}(\mathcal{D}_d, \mathcal{D}_e) = 0$ for $d > e$, or equivalently that $\operatorname{RHom}_{\mathcal{D}}(\bigwedge^{\alpha'} \mathcal{Q}, \bigwedge^{\beta'} \mathcal{Q}) = 0$ for $|\alpha| = d$, $|\beta| = e$. This follows from Proposition 3.2. □

Remark 5.7. Let $\operatorname{rep}_e(G_1)$ be the category of finite-dimensional G_1-representations of degree e. Then $\mathcal{D}_d$ is generated by $\mathcal{L}_{\mathbb{G}}(U)$ for $U \in \operatorname{rep}_d(G_1)$. Indeed, by Corollary 5.5 it suffices to show that $\mathcal{L}_{\mathbb{G}}(U)$ lies indeed in $\mathcal{D}_d$ for $U \in \operatorname{rep}_e(G_1)$ with $e \leqslant d$. To prove this we have to show that $\operatorname{Hom}(\mathcal{D}_f, \mathcal{L}_{\mathbb{G}}(U)) = 0$ for $f > d$. Given that $\mathcal{D}_f$ is generated by $\bigwedge^{\alpha'} \mathcal{Q}$ for $|\alpha| = f$ this follows again from Proposition 3.2.

The theorems we have stated have dual versions where $\mathcal{Q}$ is replaced by $\mathcal{R}$ and $B_{l,m-l}$ by $B_{m-l,l}$. We prove these by passing to the dual Grassmannian $\operatorname{Grass}(m-l, F^\vee)$.

Lemma 5.8. *The vector bundle*

$$\mathcal{T}' = \bigoplus_{\alpha \in B_{m-l,l}} \bigwedge\nolimits^{\alpha'} \mathcal{R} \tag{5.3}$$

is a tilting bundle on $\mathbb{G}$.

Proof. Using the duality $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(-, \mathcal{O}_{\mathbb{G}})$ on $\mathcal{D}$ it is sufficient to show that $\mathcal{T}'^\vee$ is a tilting bundle. Now $\mathcal{T}'^\vee$ is equal to $\bigoplus_{\alpha \in B_{m-l,l}} \bigwedge^{\alpha'} (\mathcal{R}^\vee)$ and we see that the latter is a tilting bundle by passing to the dual Grassmannian (which replaces $\mathcal{R}^\vee$ by $\mathcal{Q}$) and invoking Theorem 1.3. □

For $d \geqslant 0$ let $\mathcal{C}'_d$ be the category of finite-dimensional $G_2 = \operatorname{GL}(m-l)$-representations whose composition factors have highest weights α satisfying $|\alpha| = d$ and $\alpha \in B_{m-l,l}$. We have the following analogue of Theorem 5.4, where

$$\Phi'_d \colon D^b(\mathcal{C}'_d) \longrightarrow \mathcal{D}$$

is defined again by $U \mapsto \mathcal{L}_{\mathbb{G}}(U)$

Theorem 5.9. *The functor Φ'_d is fully faithful.*

Proof. This follows by dualizing the proof of Theorem 5.4. □

Below we let $\mathcal{D}'_d$ be the essential image of $D^b(\mathcal{C}'_d)$ under Φ'_d. We obtain the following analogue of Corollary 5.5.

Lemma 5.10. *$\mathcal{D}'_d$ is generated by $\bigwedge^{\alpha'}\mathcal{R}$ for $\alpha \in B_{m-l,l}$, $|\alpha| = d$.* □

Theorem 5.11. *The triangulated category $\mathcal{D}$ has a semi-orthogonal decomposition*

$$\left\langle \mathcal{D}'_{l(m-l)}, \dots, \mathcal{D}'_0 \right\rangle .$$

Furthermore $\mathcal{D}'_d$ is generated by $K^\alpha\mathcal{R}$ for $\alpha \in B_{m-l,l}$ with $|\alpha| = d$.

Proof. This follows by dualizing the proof of Theorem 5.6. □

Remark 5.12. As in Remark 5.7, $\mathcal{D}'_d$ is generated by $\mathcal{L}_{\mathbb{G}}(U)$ for $U \in \operatorname{rep}_d(G_2)$.

The following result finishes this section.

Theorem 5.13. *The semi-orthogonal decompositions*

$$\mathcal{D} = \left\langle \mathcal{D}_0, \dots, \mathcal{D}_{l(m-l)} \right\rangle \quad \textit{and} \quad \mathcal{D} = \left\langle \mathcal{D}'_{l(m-l)}, \dots, \mathcal{D}'_0 \right\rangle$$

are dual to each other. Furthermore the induced equivalence $\gamma_d\colon \mathcal{D}_d \longrightarrow \mathcal{D}'_d$ defined in Lemma 4.3 sends $L^\alpha\mathcal{Q}$ to $K^{\alpha'}\mathcal{R}[d]$ for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$.

Proof. To prove that the semi-orthogonal decompositions are dual, according to §4 we have to show that

$$\mathcal{D}_{\leqslant d} = \mathcal{D}'_{\leqslant d}\,,$$

where we set $\mathcal{D}_{\leqslant d} = \langle \mathcal{D}_0, \dots, \mathcal{D}_d \rangle$ and $\mathcal{D}'_{\leqslant d} = \langle \mathcal{D}'_d, \dots, \mathcal{D}'_0 \rangle$. We prove the inclusion $\mathcal{D}_{\leqslant d} \subset \mathcal{D}'_{\leqslant d}$. The opposite inclusion is similar.

From Theorem 5.6 we obtain that $\mathcal{D}_{\leqslant d}$ is generated by $L^\alpha\mathcal{Q}$ for $|\alpha| \leqslant d$. Thus we have to show that for such α we have $L^\alpha\mathcal{Q} \in \mathcal{D}'_{\leqslant d}$.

According to [Wey03, Ch 2, Ex. 21] we have a resolution for $L^\alpha\mathcal{Q}$ given by the Schur complex

$$L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})$$

and furthermore by [Wey03, Thm. (2.4.10)(b)] $L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})$ has a filtration such that

$$\operatorname{gr} L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})_t = \bigoplus_{|\nu|=t,\nu\subset\alpha} K^{\nu'}\mathcal{R} \otimes L^{\alpha/\nu}(F^\vee)\,. \tag{5.4}$$

By Remark 5.12 all $K^{\nu'}\mathcal{R}$ are in $\mathcal{D}'_{\leqslant d}$. Hence so is $L^\alpha Q$.

Assume now $|\alpha| = d$. In that case (5.4) shows that

$$L^\alpha\mathcal{Q} = K^{\alpha'}\mathcal{R}[|\alpha|] \mod \mathcal{D}'_{\leqslant d-1}\,.$$

If in addition $\alpha \in B_{l.m-l}$ then Lemma 5.10 implies $K^{\alpha'}\mathcal{R}[|\alpha|] \in \mathcal{D}'_d$, from which we conclude that $\gamma_d(L^\alpha \mathcal{Q}) = K^{\alpha'}\mathcal{R}[|\alpha|]$. □

6. Some more comments on representation theory

If we combine Theorems 5.4, 5.9, and 5.13 we obtain an equivalence of categories $\delta_d \overset{\text{def}}{=} (\Phi_d'^{-1} \circ \gamma_d \circ \Phi_d)[-d]$ between $D^b(\mathcal{C}_d)$ and $D^b(\mathcal{C}'_d)$. The existence of such an equivalence is well-known (see e.g. [Don93, Cor (3.9)] for a similar result) but the standard construction uses the representation theory of the symmetric group.

Below we list some properties of the equivalence, which we will use in §8. For α a partition let M^α be the indecomposable tilting G_1-representation with highest weight α. Similarly for $\beta \in B_{m-l,l}$ let Σ'^β be the simple G_2-representation with highest weight β and let P'^β be the projective cover of Σ'^β in $\mathcal{C}'_d$.

Since M^α has highest weight α and since $B_{l,m-l}$ is a saturated set of partitions, all the weights of M^α are in $B_{l,m-l}$, whence $M^\alpha \in \mathcal{C}_d$ by [Jan03, Lemma E.3].

Proposition 6.1. *We have for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$*

$$\delta_d(L^\alpha V) = K^{\alpha'} W \tag{6.1}$$

$$\delta_d(M^\alpha) = P'^{\alpha'}\,. \tag{6.2}$$

Proof. Statement (6.1) follows from Theorem 5.13. To prove (6.2) we first note that by suitably filtering M^α and invoking (6.1) we obtain that $\delta_d(M^\alpha) \in \mathcal{C}'_d$. Furthermore since δ_d is an equivalence for $i > 0$ we have

(6.3)
$$\operatorname{Ext}^i_{G_2}(\delta_d(M^\alpha), K^{\beta'}W) = \operatorname{Ext}^i_{G_2}(\delta_d(M^\alpha), \delta_d(L^\beta V)) = \operatorname{Ext}^i_{G_1}(M^\alpha, L^\beta V) = 0\,.$$

We now claim that we have for $i > 0$

$$\operatorname{Ext}^i_{G_2}(\delta_d(M^\alpha), \Sigma^{\beta'}) = 0\,. \tag{6.4}$$

We prove this by induction. As before we have a short exact sequence

$$0 \longrightarrow U \longrightarrow K^{\beta'}W \longrightarrow \Sigma^{\beta'} \longrightarrow 0$$

where U is obtained through extensions involving only Σ^γ with $\gamma < \beta'$. By induction we may assume $\operatorname{Ext}^i_{G_2}(\delta_d(M^\alpha), U) = 0$. Then (6.4) follows from (6.3). We conclude that $\delta_d(M^\alpha)$ is projective. Since M^α is indecomposable, the same is true for $\delta_d(M^\alpha)$. Hence $\delta_d(M^\alpha)$ is equal to some P'^γ. To prove that $\delta_d(M^\alpha) = P'^{\alpha'}$ it is sufficient to construct a surjective map

$$\delta_d(M^\alpha) \longrightarrow K^{\alpha'}W$$

since $K^{\alpha'}W$ has simple top $\Sigma'^{\alpha'}$.

By [Jan03, §E.4] we have a short exact sequence

$$0 \longrightarrow H \longrightarrow M^\alpha \longrightarrow L^\alpha V \longrightarrow 0$$

where H is an extension of $L^\gamma W$ with $\gamma < \alpha$. After applying δ_d this becomes a distinguished triangle

$$\delta_d(H) \longrightarrow \delta_d(M^\alpha) \longrightarrow K^{\alpha'}W \longrightarrow$$

with $\delta_d(H), \delta_d(M^\alpha) \in \mathcal{C}'_d$. Taking cohomology we see that $\delta_d(M^\alpha) \longrightarrow K^{\alpha'}W$ is indeed surjective. □

7. Exceptional sequences on Grassmannians

Proposition 7.1. *Assume $\alpha, \beta \in B_{l,m-l}$ with $|\alpha| = |\beta|$. If*

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{Q}, L^\beta\mathcal{Q}) \neq 0 \tag{7.1}$$

then $\alpha \geqslant \beta$. Furthermore we also have

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{Q}, L^\alpha\mathcal{Q}) = K\,.$$

Proof. We have $L^\alpha\mathcal{Q} = \Phi_d(L^\alpha V)$, $L^\beta\mathcal{Q} = \Phi_d(L^\beta V)$. So to prove the first claim, by Theorem 5.9 we must show that

$$\operatorname{RHom}_{G_1}(L^\alpha V, L^\beta V) \neq 0$$

implies $\alpha \geqslant \beta$. Since $L^\alpha V$, $L^\beta V$ are induced representations it suffices to invoke [Jan03, Prop. II.6.20].

By [Jan03, Prop. II.2.8] we have $\operatorname{Hom}_{G_1}(L^\alpha V, L^\alpha V) = K$. Hence to prove the second claim we have to show

$$\operatorname{Ext}^i_{G_1}(L^\alpha V, L^\alpha V) = 0$$

for $i > 0$. This follows from [Jan03, Prop. II.6.20]. □

Proposition 7.2. *Assume $\alpha, \beta \in B_{m-l,l}$, $|\alpha| = |\beta|$. Then*

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{R}, L^\beta\mathcal{R}) \neq 0 \tag{7.2}$$

implies $\alpha \geqslant \beta$. Furthermore we also have

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{R}, L^\alpha\mathcal{R}) = K\,.$$

Proof. This is proved in exactly the same way as Proposition 7.1. □

Proposition 7.3. *For $\alpha, \beta \in B_{m-l,l}$, $|\alpha| = |\beta|$ we have*

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(K^\alpha\mathcal{R}, L^\beta\mathcal{R}) = \delta_{\alpha,\beta} \cdot K\,.$$

Proof. Put $d = |\alpha| = |\beta|$. We have $K^\alpha\mathcal{R} = \Phi'_d(K^\alpha W)$, $L^\beta\mathcal{R} = \Phi'_d(L^\beta W)$. So by Theorem 5.9 we must show

$$\operatorname{RHom}_{G_2}(K^\alpha W, L^\beta W) = \delta_{\alpha,\beta}\cdot K\,.$$

As $K^\alpha W$ is a Weyl representation and $L^\beta W$ is an induced representation, it suffices to invoke [Jan03, II.4.13]. □

Now we make $B_{l,m-l}$ into a totally ordered set by equipping it with an arbitrary total ordering $\prec$ such that if $|\alpha| < |\beta|$ then $\alpha \prec \beta$ and if $|\alpha| = |\beta|$ and $\alpha > \beta$ in the standard partial order on partitions, then $\alpha \prec \beta$. We write $\overline{B}_{l,m-l}$ for $B_{l,m-l}$, equipped with the opposite ordering.

The following is the main result of this section.

Theorem 7.4. *The collections $(L^\alpha\mathcal{Q})_{\alpha\in B_{l,m-l}}$ and $(L^{\beta'}\mathcal{R}[|\beta|])_{\beta\in\overline{B}_{l,m-l}}$ form dual exceptional collections in $\mathcal{D}$. In other words for $\alpha,\beta\in B_{l,m-l}$ we have*

$$\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{Q}, L^{\beta'}\mathcal{R}[|\beta|]) = \delta_{\alpha,\beta}\cdot K\,. \tag{7.3}$$

Proof. The fact that $(L^\alpha\mathcal{Q})_{\alpha\in B_{l,m-l}}$ is an exceptional sequence follows from Theorem 5.6 and Proposition 7.1. Similarly the fact that $(L^{\beta'}\mathcal{R}[|\beta|])_{\beta\in\overline{B}_{l,m-l}}$ is an exceptional collection follows from Theorem 5.11 and Proposition 7.2. So it remains to prove the duality property (7.3). By Theorem 5.13 combined with (4.1) we may assume that $|\alpha| = |\beta|$. We compute

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(L^\alpha\mathcal{Q}, L^{\beta'}\mathcal{R}[|\beta|]) &= \operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\gamma_i(L^\alpha\mathcal{Q}), L^{\beta'}\mathcal{R}[|\beta|])\\ &= \operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(K^{\alpha'}\mathcal{R}[|\alpha|], L^{\beta'}\mathcal{R}[|\beta|])\\ &= \delta_{\alpha,\beta}\cdot K\,,\end{aligned}$$

using, respectively, Lemma 4.3, Theorem 5.13 and Lemma 4.3, and Proposition 7.3. □

To conclude this section we use the "linkage principle" [Jan03, Cor. II.6.17] to recover the result of Kaneda, mentioned in the Introduction, that Kapranov's tilting bundle $\mathcal{K}$ remains tilting in large characteristic.

Lemma 7.5. *Assume that K has characteristic p with $p \geqslant m-1$. Let $\alpha\in B_{l,m-l}$. Then $\bigwedge^{\alpha'}\mathcal{Q}$ is a direct sum of $L^\beta\mathcal{Q}$ with $|\beta| = |\alpha|$ and furthermore there are no homomorphisms between the summands of $\bigwedge^{\alpha'}\mathcal{Q}$.*

Proof. Set $d = |\alpha|$. Using Theorem 5.4 it is enough to prove the following claim: $\mathcal{C}_d$ is a semi-simple category with simple objects given by $L^\beta V$ for $\beta\in B_{l,m-l}$ with $|\beta| = d$. Indeed if this claim holds then

$\bigwedge^{\alpha'} V$ is a direct sum of the simple objects $L^\beta V$ (which thus have no Hom's among them) and it suffices to apply the fully faithful functor $\Phi(-)_d$ to obtain the corresponding result for $\bigwedge^{\alpha'} \mathcal{Q}$.

The claim follows directly from the linkage principle [Jan03, Cor. II.6.17] which we state in the case of interest to us. If γ, δ are dominant weights for G_1 and $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{G}}}(\Sigma^\gamma, \Sigma^\delta) \neq 0$ then γ, δ are in the same orbit for the *affine Weyl group*.

A fundamental domain[2] $\overline{C}$ for the affine Weyl group ([Jan03, II.6.1(6)]) is given by the set of $x = [x_1, \ldots, x_l]$ satisfying

$$0 \leqslant x_i - i - x_j + j \leqslant p$$

for $j > i$. The first inequality is automatically satisfied for a dominant weight. For the second inequality we note that if $\gamma = [\gamma_1, \ldots, \gamma_l] \in B_{l,m-l}$ then

$$\gamma_i - \gamma_j \leqslant m - l$$

and

$$-i + j \leqslant l - 1\,.$$

Thus

$$\gamma_i - i - \gamma_j + j \leqslant m - l + l - 1 = m - 1 \leqslant p\,.$$

In other words $B_{l,m-l} \subseteq \overline{C}$ and thus no two elements of $B_{l,m-l}$ are in the same orbit for the affine Weyl group. The claim follows. $\square$

The fact that $\bigoplus_{\alpha \in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}$ is a tilting object, together with Theorem 5.6 and the previous lemma, yields immediately the following.

Corollary 7.6 (Kaneda [Kan08])**.** *The Kapranov strong exceptional collection $(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ remains strong exceptional as long as $p \geqslant m-1$. In particular $\mathcal{K} = \bigoplus_{\alpha \in B_{l,m-l}} L^\alpha \mathcal{Q}$ remains tilting for such p.*

8. Relation with quasi-hereditary algebras

We quickly remind the reader of the module-theoretic description of quasi-hereditary algebras à la Dlab-Ringel [DR92]. Let A be a finite-dimensional K-algebra and let $(S(\lambda))_{\lambda \in \Lambda}$ be a complete set of the simples, with projective covers $P(\lambda)$ and injective hulls $Q(\lambda)$. Given some total ordering $\prec$ on Λ, we define the *standard module* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ having composition factors of the form $S(\mu)$ with $\mu \preceq \lambda$. Similarly the *costandard module* $\nabla(\lambda)$ is the largest submodule of $Q(\lambda)$ having composition factors $S(\mu)$ with $\mu \preceq \lambda$. Assume that $\operatorname{End}_A(\Delta(\lambda))$ is a division ring for each λ. Then A (with

[2]A fundamental domain is a complete irredundant set of orbit representatives [Bou02, IV.3.3].

the fixed order $\prec$) is called *quasi-hereditary* if ${}_AA$ can be filtered by standard modules; equivalently, the sets $\{X \mid \operatorname{Ext}^1_A(X, \nabla(\lambda)) = 0\}$ and $\{X \mid \operatorname{Ext}^1_A(\nabla(\lambda), X) = 0\}$ are equal and coincide with the set of modules filtered by the standard modules.

As in the introduction put

$$\mathcal{T} = \bigoplus_{\alpha \in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}$$

and $A = \operatorname{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T})$. Denote by A° the opposite algebra. For $\alpha \in B_{l,m-l}$ consider the following complexes of right A-modules.

$$\Delta(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^\alpha \mathcal{Q}) \tag{8.1}$$

$$\nabla(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^{\alpha'} \mathcal{R}[|\alpha|]) \tag{8.2}$$

$$P(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{L}(M^\alpha)) \tag{8.3}$$

$$S(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{L}(\Sigma'^{\alpha'})[|\alpha|]) \tag{8.4}$$

Theorem 8.1. *The complexes $\Delta(\alpha)$, $\nabla(\alpha)$, $P(\alpha)$, $S(\alpha)$ are concentrated in degree zero and furthermore the $S(\alpha)$ are the simple right A-modules with the $P(\alpha)$ their projective covers. Moreover if we order $B_{l,m-l}$ using $\prec$ as above then A is quasi-hereditary and the standard and costandard modules having $S(\alpha)$ respectively as top and socle are $\Delta(\alpha)$ and $\nabla(\alpha)$.*

Proof. Since $\Theta = (L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ is exceptional by Theorem 7.4, it follows in particular that the collection Θ is *standardizable* in the sense of [DR92, §3]. By the proof of [DR92, Theorem 2] there exists a projective generator P in the exact category $\mathcal{F}(\Theta)$, which consists of repeated extensions of objects in Θ, such that $A' = \operatorname{End}(P)$ is quasi-hereditary and such that the objects in Θ correspond to the standard objects in $\operatorname{Mod}(A')$ using the functor $\operatorname{Hom}(P', -)$.

On the other hand Proposition 1.4 implies that $\mathcal{T}$ is a projective generator for $\mathcal{F}(\Theta)$ as well. This easily yields that A and A' are Morita equivalent and that the objects $\operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^\alpha \mathcal{Q})$ are the standard objects. By Proposition 1.4 we may replace Hom by RHom.

The costandard modules $\nabla(\beta)$ in $D^b_f(A^\circ)$ (the bounded derived category with finite cohomology) are characterized by

$$\operatorname{RHom}_{A^\circ}(\Delta(\alpha), \nabla(\beta)) = \delta_{\alpha,\beta} \cdot K\,. \tag{8.5}$$

From (7.3) we then deduce that they are indeed given by the formula (8.2).

By [Jan03, Lemma E.5] $\bigwedge^{\alpha'} V$ is a direct sum of various M^β, and M^α occurs as a direct summand with multiplicity one in $\bigwedge^{\alpha'} V$. It follows that the $P(\alpha)$ are indeed the indecomposable projectives.

To show that the $S(\alpha)$ are the corresponding simple A-modules we have to prove

$$\mathrm{RHom}_{A^\circ}(P(\alpha), S(\beta)) = \delta_{\alpha,\beta} \cdot K\,.$$

We compute

$$\mathrm{RHom}_{A^\circ}(P(\alpha), S(\beta)) = \mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{L}(M^\alpha), \mathcal{L}(\Sigma'^{\beta'})[|\beta|])\,.$$

It follows from Theorem 5.11 combined with (4.1) that if $|\alpha| \neq |\beta|$ then there is nothing to prove. So assume $|\alpha| = |\beta| = d$. Then we have

$$\begin{aligned}
\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{L}(M^\alpha), \mathcal{L}(\Sigma'^{\beta'})[|\beta|]) &= \mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\gamma_d(\Phi_d(M^\alpha)), \Phi'_d(\Sigma'^{\beta'})[|\beta|]) \\
&= \mathrm{RHom}_{G_2}(\delta_d(M^\alpha), \Sigma'^{\beta'}) \\
&= \mathrm{RHom}_{G_2}(P'^{\alpha'}, \Sigma'^{\beta'}) \\
&= \delta_{\alpha,\beta} \cdot K\,,
\end{aligned}$$

where the first equality is Lemma 4.3 and the third is Proposition 6.1; in the second, δ_d is as introduced in §6. It remains to prove that $S(\alpha)$ is the top of $\Delta(\alpha)$ and the socle of $\nabla(\alpha)$. Since for quasi-hereditary algebras the top of $\Delta(\alpha)$ coincides with the socle of $\nabla(\alpha)$ it is sufficient to prove only the first of these statements.

Since there is a surjective map $M^\alpha \longrightarrow L^\alpha V$ whose kernel is an extension of $L^\beta V$ with $\beta < \alpha$ [Jan03, §E.4] we obtain using Proposition 1.4 a surjective map $P(\alpha) \longrightarrow \Delta(\alpha)$. This finishes the proof. $\square$

Example 8.2. We compute the quiver and relations of the quasi-hereditary algebra A in the first non-trivial example, $(m, l) = (4, 2)$. We live inside the 2×2 box $B_{2,2}$, so the vertices of the quiver, equivalently the summands of the tilting bundle $\mathcal{T}$, are labeled

$$\mathcal{O},\ \mathcal{Q},\ \textstyle\bigwedge^2\mathcal{Q},\ \mathcal{Q}\otimes\mathcal{Q},\ \bigwedge^2\mathcal{Q}\otimes\mathcal{Q},\ \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2}\,.$$

The quiver has the following form.

$$\begin{tikzcd}
 & & \bigwedge^2\mathcal{Q} \arrow[dd, shift left, "a"] & & \\
\mathcal{O} \arrow[r, "s_\lambda"] & \mathcal{Q} \arrow[dr, "\alpha_\lambda"'] & & \bigwedge^2\mathcal{Q}\otimes\mathcal{Q} \arrow[r, "t_\lambda"] & \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2} \\
 & & \mathcal{Q}\otimes\mathcal{Q} \arrow[uu, shift left, "p"] \arrow[ur, "\beta_\lambda"'] & &
\end{tikzcd}$$

The labels stand for natural maps between these bundles, some of which depend on a global section $\lambda \in F^\vee$:

- $p\colon \mathcal{Q}\otimes\mathcal{Q} \longrightarrow \bigwedge^2\mathcal{Q}$ the natural surjection and $a\colon \bigwedge^2\mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ the anti-symmetrization;
- $s_\lambda\colon \mathcal{O} \longrightarrow \mathcal{Q}$ with $1 \mapsto \lambda$;
- $\alpha_\lambda\colon \mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ with $x \mapsto \lambda\otimes x\,$;
- $\beta_\lambda\colon \mathcal{Q}\otimes\mathcal{Q} \longrightarrow \bigwedge^2\mathcal{Q}\otimes\mathcal{Q}$ with $x\otimes y \mapsto \lambda\wedge y\otimes x\,$; and
- $t_\lambda\colon \bigwedge^2\mathcal{Q}\otimes\mathcal{Q} \longrightarrow \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2}$ with $x\wedge y\otimes z \mapsto x\wedge y\otimes\lambda\wedge z$.

These maps generate all the arrows in the quiver. For example, the obvious complementary map $\alpha'_\lambda\colon \mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ defined by $\alpha'_\lambda(x) = x\otimes\lambda$ can be obtained as $\alpha_\lambda(1-ap)$. The relations are most compactly written in terms of the pseudo-idempotent $e \overset{\text{def}}{=} ap$ satisfying $e^2 = 2e$, and the "swap" $1-e$ which sends $x\otimes y$ to $y\otimes x$. We have

- $pa = 2\,\mathrm{Id}_{\bigwedge^2\mathcal{Q}}\,$;
- $(1-e)\alpha_\lambda s_\mu = \alpha_\mu s_\lambda\,$;
- $\beta_\lambda(1-e)\alpha_\mu = \beta_\lambda\alpha_\mu - \beta_\mu\alpha_\lambda\,$;
- $t_\lambda\beta_\mu(1-e) = t_\mu\beta_\lambda\,$.

We observe that in this picture, each vertical "slice" is equivalent to the derived category of a generalized Schur algebra. For example, in the middle we recognize the quiver for the Schur algebra $S(2,2)$ in characteristic 2 [Erd93, 3.1.1, 5.4].

In characteristic different from 2, the idempotent $\frac{1}{2}e$ gives $\mathcal{Q}\otimes\mathcal{Q} \cong \bigwedge^2\mathcal{Q}\oplus\mathrm{Sym}_2\mathcal{Q}$, and the algebra becomes Morita-equivalent to the path algebra of the equivariant quiver

$$\begin{tikzcd}
& & \bigwedge^2\mathcal{Q} \arrow[dr, "F^\vee"] \arrow[drr, dashed, "D_2F^\vee"] & & \\
\mathcal{O} \arrow[r, "F^\vee"] \arrow[urr, dashed, "D_2F^\vee"] \arrow[drr, dashed, "\bigwedge^2 F^\vee"'] & \mathcal{Q} \arrow[ur, "F^\vee"] \arrow[dr, "F^\vee"'] \arrow[rr, dashed, "F^\vee\otimes F^\vee"] & & \bigwedge^2\mathcal{Q}\otimes\mathcal{Q} \arrow[r, "F^\vee"] & \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2} \\
& & \mathrm{Sym}_2\mathcal{Q} \arrow[ur, "F^\vee"'] \arrow[urr, dashed, "\bigwedge^2 F^\vee"'] & &
\end{tikzcd}$$

with relations

- $\mathcal{O} \longrightarrow \bigwedge^2\mathcal{Q}$ given by $D_2F^\vee$;
- $\mathcal{O} \longrightarrow \mathrm{Sym}_2\mathcal{Q}$ given by $\bigwedge^2 F^\vee$;
- $\mathcal{Q} \longrightarrow \bigwedge^2\mathcal{Q}\otimes\mathcal{Q}$ given by $F^\vee\otimes F^\vee$;
- $\bigwedge^2\mathcal{Q} \longrightarrow \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2}$ given by $D_2F^\vee$; and
- $\mathrm{Sym}_2\mathcal{Q} \longrightarrow \left(\bigwedge^2\mathcal{Q}\right)^{\otimes 2}$ given by $\bigwedge^2 F^\vee$.

Most of these are straightforward to verify. The relations across the central diamond, however, are not the obvious commutativity ones [Hil98]. To compute those relations, give names to the maps:

$$\begin{tikzcd} & \bigwedge^2 \mathcal{Q} \arrow[dr, "b_\lambda"] & \\ \mathcal{Q} \arrow[ur, "a_\lambda"] \arrow[dr, "c_\lambda"'] & & \bigwedge^2 \mathcal{Q} \otimes \mathcal{Q} \\ & \operatorname{Sym}_2 \mathcal{Q} \arrow[ur, "d_\lambda"'] & \end{tikzcd}$$

with

- $a_\lambda(x) = \lambda \wedge x$;
- $b_\lambda(x \wedge y) = x \wedge y \otimes \lambda$;
- $c_\lambda(x) = \lambda x$; and
- $d_\lambda(xy) = \lambda \wedge x \otimes y + \lambda \wedge y \otimes x$.

Then we find, for $\lambda, \mu \in F^\vee$,

$$d_\mu c_\lambda = 2 b_\lambda a_\mu - b_\mu a_\lambda \,.$$

It follows that the defining relations are

$$\begin{aligned} d_\lambda c_\mu + d_\mu c_\lambda &= b_\lambda a_\mu + b_\mu a_\lambda \\ d_\mu c_\lambda - d_\lambda c_\mu &= 3(b_\lambda a_\mu - b_\mu a_\lambda)\,. \end{aligned}$$

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E-mail address: `ragnar@utsc.utoronto.ca`

Dept. of Computer and Mathematical Sciences, University of Toronto Scarborough, Toronto, Ont. M1C 1A4, Canada

E-mail address: `gjleusch@math.syr.edu`

Dept. of Mathematics, Syracuse University, Syracuse NY 13244, USA
URL: `http://www.leuschke.org/`

E-mail address: `michel.vandenbergh@uhasselt.be`

Departement WNI, Universiteit Hasselt, 3590 Diepenbeek, Belgium