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Hypersurfaces of bounded Cohen–Macaulay type[☆]

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Dedicated to Prof. Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

Let $R = k[[x_0, \dots, x_d]]/(f)$, where k is a field and f is a non-zero non-unit of the formal power series ring $k[[x_0, \dots, x_d]]$. We investigate the question of which rings of this form have bounded Cohen–Macaulay type, that is, have a bound on the multiplicities of the indecomposable maximal Cohen–Macaulay modules. As with *finite* Cohen–Macaulay type, if the characteristic is different from two, the question reduces to the one-dimensional case: The ring R has bounded Cohen–Macaulay type if and only if $R \cong k[[x_0, \dots, x_d]]/(g + x_2^2 + \dots + x_d^2)$, where $g \in k[[x_0, x_1]]$ and $k[[x_0, x_1]]/(g)$ has bounded Cohen–Macaulay type. We determine which rings of the form $k[[x_0, x_1]]/(g)$ have bounded Cohen–Macaulay type.

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0. Introduction

Throughout this paper (R, \mathfrak{m}, k) will denote a Cohen–Macaulay local ring (with maximal ideal \mathfrak{m} and residue field k). A maximal Cohen–Macaulay R -module (MCM module for short) is a finitely generated R -module with $\text{depth}(M) = \dim(R)$. We say that R has *bounded* Cohen–Macaulay (CM) type provided there is a bound on the multiplicities of

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the indecomposable MCM modules. One goal of this paper is to examine the distinction between this property and the formally stronger property of *finite* CM type—that there exist, up to isomorphism, only finitely many indecomposable MCM modules.

We denote the multiplicity of a finitely generated module M by $e(M)$. Following Scheja and Storch [24], we say that M has a rank provided $K \otimes_R M$ is K -free, where K is the total quotient ring of R (obtained by inverting all non-zero-divisors). If $K \otimes_R M \cong K^r$ we say $\text{rank}(M) = r$. In this case $e(M) = r \cdot e(R)$, [2, (4.6.9)], so for modules with rank, a bound on multiplicities is equivalent to a bound on ranks.

The one-dimensional CM local rings of finite CM type have been completely characterized. To state the characterization, we let \widehat{R} be the \mathfrak{m} -adic completion of R and \overline{R} the integral closure of R in its total quotient ring K .

Theorem 0.1. *Let (R, \mathfrak{m}) be a one-dimensional CM local ring. The following are equivalent:*

- (1) R has finite CM type.
- (2) R has bounded CM type and \widehat{R} is reduced.
- (3) R satisfies the “Drozd–Roĭter conditions”:

- (dr1) $e(R) \leq 3$; and
- (dr2) $\frac{\mathfrak{m}\overline{R} + \overline{R}}{R}$ is cyclic as an R -module.

This result was asserted (in different but equivalent form) in a 1967 paper [9] by Drozd and Roĭter, and they sketched a proof in the “arithmetic” case, where R is a localization of a module-finite \mathbb{Z} -algebra. Their proof that (2) \Rightarrow (3) goes through in the general case, but their proof that (3) \Rightarrow (1) is rather obscure even in the arithmetic case. In 1978 Green and Reiner [10] gave detailed matrix reductions proving that (3) \Rightarrow (1) in the arithmetic case. In [25] Wiegand used the approach in [10] to prove the theorem under the additional hypothesis that the residue field R/\mathfrak{m} is perfect. Later, in [26], he showed that the theorem is true as long as the residue field does not have characteristic 2. Finally, in his 1994 Ph.D. dissertation [4], Çimen completed the intricate matrix reductions necessary to prove the theorem in general. In [6] one can find a streamlined proof of everything but the matrix reductions, which appear in [5]. (A word about the implication (1) \Rightarrow (2) is in order. If R has finite CM type, so has the completion \widehat{R} , [26, Corollary 2]. Now by [26, Proposition 1] \widehat{R} is reduced. Also, we mention that condition (dr2) in (3) implies that \overline{R} is finitely generated as an R -module; therefore (dr1) could be replaced by the condition that \overline{R} can be generated by three elements as an R -module.)

Thus, for one-dimensional analytically unramified local CM rings, finite CM type and bounded CM type are equivalent. A statement of this form—that bounded representation type implies finite representation type—is often called the “first Brauer–Thrall conjecture”; see [19] for some history on this and related conjectures. In particular, the statement for finite-dimensional algebras over a field is a theorem due to Roĭter [20]. The first example showing that the two concepts are not equivalent in the context of MCM modules was given by Dieterich in 1980 [8]: Let k be a field of characteristic 2, let $A = k[[x]]$, and let G be the two-element group. Then the group ring AG has bounded CM type. Note that

$AG \cong k[[x, y]]/(y^2)$. Thus condition (2) of Theorem 0.1 fails, and AG has infinite CM type. Later Buchweitz et al. [3] noted that $k[[x, y]]/(y^2)$ has bounded CM type for every field k . In Section 2 we will show, by adapting an argument due to Bass [1], that every one-dimensional CM ring of multiplicity 2 has bounded CM type.

In Section 1 we show that, for complete equicharacteristic hypersurfaces, the question of bounded CM type reduces to the case of plane curve singularities, and in Section 2 we examine that case. We are able to answer the question completely: A one-dimensional complete equicharacteristic hypersurface R has bounded CM type if and only if either (a) R has finite CM type, (b) $R \cong k[[x, y]]/(y^2)$ or (c) $R \cong k[[x, y]]/(xy^2)$. The indecomposable MCM modules over $k[[x, y]]/(xy^2)$ were classified by Buchweitz et al. ([3], see also Theorem 2.8 below). When $k = \mathbb{C}$, the field of complex numbers, the rings in (b) and (c) are the exactly the rings of countably infinite CM type discussed in [3]. They are the limiting cases A_∞ and D_∞ of the A_n and D_n singularities $k[[x, y]]/(x^{n+1} + y^2)$ and $k[[x, y]]/(x^{n-1} + xy^2)$, both of which have finite CM type. We note that the families of ideals exhibiting uncountable deformation type in (3.5) of [3] do not give rise to indecomposable modules of large rank; thus there does not seem to be a way to use the results of [3] to demonstrate unbounded CM type in the cases not covered by (a)–(c).

1. Complete equicharacteristic hypersurfaces of dimension two or more

Our goal in this section is to prove the following analog, for bounded CM type, of a beautiful result of Buchweitz, Greuel, Knörrer and Schreyer [3,17] on finite CM type:

Theorem 1.1. *Let k be a field, and let $R = k[[x_0, \dots, x_d]]/(f)$, where f is a non-zero non-unit of the formal power series ring $k[[x_0, \dots, x_d]]$, $d \geq 2$. Assume that the characteristic of k is different from 2. Then R has bounded CM type if and only if $R \cong k[[x_0, \dots, x_d]]/(g + x_2^2 + \dots + x_d^2)$, for some $g \in k[[x_0, x_1]]$ such that $k[[x_0, x_1]]/(g)$ has bounded CM type.*

1.1. The double branched cover

We first set some notation. In this section, we will set $S := k[[x_0, \dots, x_d]]$, a ring of formal power series over a field k , and we denote its maximal ideal by \mathfrak{n} . We fix a non-zero element $f \in \mathfrak{n}^2$ and set $R = S/(f)$. We define the *double branched cover* R^\sharp of R by $R^\sharp = S[[z]]/(f + z^2)$, where z is a new indeterminate over S . Note that there is a natural surjection $R^\sharp \rightarrow R$ defined by $\bar{z} \mapsto 0$, where \bar{z} is the coset of z in R^\sharp . There are functors from the category of MCM R -modules to that of MCM R^\sharp -modules, and inversely, defined as follows. For a MCM R -module M , set $M^\sharp = \text{syz}_{R^\sharp}^1(M)$, and for a MCM R^\sharp -module N , set $\bar{N} = N/zN$. We have the following relation on the compositions of these two functors:

Proposition 1.2 ([28, Proposition 12.4]). *With notation as above, assume that M has no non-zero free summand. Then $\overline{M^\sharp} \cong M \oplus \text{syz}_R^1(M)$. If $\text{char}(k) \neq 2$, then $\bar{N}^\sharp \cong N \oplus \text{syz}_{R^\sharp}^1(N)$.*

(The restriction on the characteristic of k does not appear in Yoshino’s version, which treats only characteristic zero, but the proof there is easily seen to apply in this context.)

This allows us to show that bounded CM type ascends to and descends from the double branched cover. We use two slightly more general lemmas, the first of which is due to Herzog, and the second of which says that for a Gorenstein ring A , a bound on the multiplicities of indecomposable MCM A -modules is equivalent to a bound on their numbers of generators. We denote by $v_R(M)$ the minimal number of generators required for M as an R -module.

Lemma 1.3 (Herzog [13, Lemma 1.3]). *Let A be a Gorenstein local ring and let M be an indecomposable non-free MCM R -module. Then $\text{syz}_A^1(M)$ is indecomposable.*

Lemma 1.4. *Let A be a Gorenstein local ring. Then there is a bound on the multiplicities of indecomposable MCM A -modules if and only if there is a bound on the number of generators of same.*

Proof. Let M be an indecomposable non-free MCM A -module, $n = v_A(M)$, and let $N = \text{syz}_A^1(M)$, so that we have the following exact sequence

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow M \longrightarrow 0.$$

By Lemma 1.3, N is also indecomposable. If the multiplicities of both M and N are bounded above by B , then $ne(A) = e(A^n) \leq 2B$, so $n \leq 2B/e(A)$. Conversely, if n is bounded by some number B , then $e(M) \leq e(A^n) \leq Be(A)$, so the multiplicity of M is bounded. \square

Proposition 1.5. *Let $R = S/(f)$ be a complete hypersurface, where $S = k[[x_0, \dots, x_d]]$ and f is a non-zero non-unit of S .*

- (1) *If R^\sharp has bounded CM type, then R has bounded CM type as well.*
- (2) *If the characteristic of k is not 2, then the converse holds as well. In fact, if $v_R(M) \leq B$ for each indecomposable MCM R -module, then $v_{R^\sharp}(N) \leq 2B$ for each indecomposable MCM R^\sharp -module N .*

Proof. To prove (1), let M be an indecomposable non-free MCM R -module. Then by Proposition 1.2 $\overline{M}^\sharp \cong M \oplus \text{syz}_R^1(M)$, so M is a direct summand of \overline{M}^\sharp . Decompose M^\sharp into indecomposable MCM R^\sharp -modules, $M^\sharp \cong N_1 \oplus \dots \oplus N_t$, where each N_i requires at most B generators. Then $\overline{M}^\sharp \cong \overline{N}_1 \oplus \dots \oplus \overline{N}_t$, and by the Krull–Schmidt uniqueness theorem M is a direct summand of some \overline{N}_j . Since $v_R(\overline{N}_j) = v_{R^\sharp}(N_j)$, the result follows.

For the converse, let N be an indecomposable non-free MCM R^\sharp -module. By Proposition 1.2, $\overline{N}^\sharp \cong N \oplus \text{syz}_{R^\sharp}^1(N)$. Decompose \overline{N} into indecomposable MCM R -modules, $\overline{N} \cong M_1 \oplus \dots \oplus M_s$, with $v_R(M_j) \leq B$ for each j . Then $\overline{N}^\sharp \cong M_1^\sharp \oplus \dots \oplus M_s^\sharp$. By the Krull–Schmidt theorem again, N is a direct summand of some M_j^\sharp . It will suffice to show that $v_{R^\sharp}(M_j^\sharp) \leq B$ for each j .

If M_j is not free, we have $v_{R^\sharp}(M_j^\sharp) = v_R(\overline{M}_j^\sharp) = v_R(M_j) + v_R(\text{syz}_R^1(M))$ by Proposition 1.2. But since M_j is a MCM R -module, all of its Betti numbers are equal to $v_R(M_j)$, [28,

(7.2.3)]. Thus $v_R(\overline{M_j^\sharp}) = 2v_R(M_j) \leq 2B$. If, on the other hand, $M_j = R$, then $M_j^\sharp = zR^\sharp \cong R^\sharp$, and $v_R(\overline{M_j^\sharp}) = 1$. \square

1.2. Multiplicity and reduction to dimension one

Our next concern is to show that a hypersurface of bounded representation type has multiplicity at most two, as long as the dimension is greater than one. This is a corollary of the following result of Kawasaki ([15], due originally in the graded case to Herzog and Sanders [14]). An *abstract hypersurface* is a Noetherian local ring (R, \mathfrak{m}) such that the \mathfrak{m} -adic completion \widehat{R} is isomorphic to $S/(f)$ for some regular local ring S and non-unit f .

Theorem 1.6 (Kawasaki [15, Theorem 4.1]). *Let (R, \mathfrak{m}) be an abstract hypersurface of dimension d . Assume that the multiplicity $e(R)$ is greater than 2. Then for each $n > e$, the maximal CM module $\text{syz}_R^{d+1}(R/\mathfrak{m}^n)$ is indecomposable and*

$$v_R(\text{syz}_R^{d+1}(R/\mathfrak{m}^n)) \geq \binom{d+n-1}{d-1}.$$

Corollary 1.7. *Let R be an abstract hypersurface with $\dim(R) > 1$ and $e(R) > 2$. Then R does not have bounded CM type.*

Proposition 1.8. *Let (R, \mathfrak{m}, k) be a Gorenstein local ring of bounded CM type. Then R is an abstract hypersurface. If R is complete, $d := \dim(R) \geq 2$, and R contains a field of characteristic not equal to 2, then either R is regular or $R \cong A^\sharp$ for some hypersurface A , which also has bounded CM type.*

Proof. To show that R is an abstract hypersurface, it suffices (as in the proof of [13, Lemma 1.2]) to show that the Betti numbers $\beta_R^n(k)$ are bounded. Let $M = \text{syz}_R^d(k)$ and decompose M into non-zero indecomposable MCM modules, $M = \bigoplus_{i=1}^t M_i$. Assume that M_1, \dots, M_s are non-free and M_{s+1}, \dots, M_t are free modules. By Lemma 1.3, each subsequent syzygy module of k has exactly s indecomposable summands, so if B is a bound for the number of generators of MCM modules, then $\beta_R^n(k) \leq sB$ for all $n \geq d$.

For the final statement, we assume that R is complete and not regular. By the first part, we can write $R \cong S/(f)$ where $S = k[[x_0, \dots, x_d]]$ is a power series ring over a field and $d \geq 2$. Write $f = \sum_{i=0}^{\infty} f_i$, where each f_i is a homogeneous polynomial in x_0, \dots, x_d of degree i . Since, by Corollary 1.7, $e(R) = 2$, we have $f_0 = f_1 = 0$ and $f_2 \neq 0$. We may assume after a linear change of variables that f_2 contains a term of the form cx_d^2 , where c is a non-zero element of k . Now consider f as a power series in one variable, x_d , over $S' := k[[x_0, \dots, x_{d-1}]]$. As such, the constant term and the coefficient of x_d are in the maximal ideal of S' . The coefficient of x_d^2 is of the form $c + g$, where g is in the maximal ideal of S' . Therefore, by [18, Theorem 9.2], f can be written uniquely in the form

$$f(x_d) = u(x_d^2 + b_1x_d + b_2),$$

where the b_i are elements of the maximal ideal of S' and u is a unit of S .

We may ignore the presence of u , as it does not change R . Then, since $\text{char}(k) \neq 2$, we can complete the square and, after a linear change of variables, write $f = x_d^2 + h(x_0, \dots, x_{d-1})$ for some power series $h \in S'$. By Proposition 1.5, $A := S'/(h)$ has bounded CM type. \square

2. Dimension one

The results of the previous section reduce our problem to the case of one-dimensional hypersurface rings. In this section we will deal with this case. Some of our results go through for more general one-dimensional CM local rings, not just hypersurfaces. We note that over a one-dimensional CM local ring the MCM modules are exactly the non-zero finitely generated torsion-free modules.

2.1. Multiplicity two

We begin with a positive result, which puts the examples of [8,3] mentioned earlier into a general context. In the analytically unramified case, the result below is due to Bass [1]. In [21] Rush proved the result in the analytically ramified case, but only for modules with rank. Here we will show how to remove this restriction.

Theorem 2.1. *Let (R, \mathfrak{m}) be a one-dimensional CM local ring with $e(R) = 2$. Then every MCM R -module is isomorphic to a direct sum of ideals of R . In particular, every indecomposable MCM R -module has multiplicity at most 2 and is generated by at most 2 elements.*

Proof. We note that every ideal of R is generated by two elements, [22, Chapter 3, Theorem 1.1]. If the integral closure \bar{R} of R in the total quotient ring K of R is finitely generated over R , the theorem follows from [1, (7.1), (7.3)]. Therefore we assume from now on that \bar{R} is not a finitely generated R -module.

Suppose S is an arbitrary module-finite extension of R contained in K . By [23, Theorem 3.6], \bar{R} is quasi-local, and it follows that S is local. Moreover, each ideal of S is isomorphic to an ideal of R and is therefore generated by two elements (as an R - or S -module). By [1, (6.4)] S is Gorenstein. In particular, R itself is Gorenstein.

To complete the proof, it will suffice to show that every MCM R -module M has a direct summand isomorphic to a non-zero ideal of R . The first part of the argument here is due to Bass [1, (7.2)]. Suppose first that M is faithful. Let $S = \{\alpha \in K \mid \alpha M \subseteq M\}$. Since M is faithful, S is a subring of $\text{Hom}_R(M, M)$ and therefore is a module-finite extension of R . Of course M is a MCM S -module. If there is a surjection $M \rightarrow S$, then M has a direct summand isomorphic to S , which, in turn, is isomorphic to an ideal of R . Therefore we suppose to the contrary that $M^* = \text{Hom}_S(M, \mathfrak{n})$, where $(_)^*$ denotes the S -dual and \mathfrak{n} is the maximal ideal of S . Now M^* is a module over $E := \text{Hom}_S(\mathfrak{n}, \mathfrak{n})$ and therefore so is M^{**} . But since S is Gorenstein, M is reflexive [1, (6.2)]. Therefore M is actually an E -module. Since $S \subseteq E \subseteq K$ we must have $S = E$ (by the definition of S). It follows easily that \mathfrak{n} is a principal ideal, that is, S is a discrete valuation ring. But then M has S as a free summand, a contradiction.

If M is not faithful, let $I = (0 : M)$. We claim that M has a direct summand isomorphic to an ideal of R/I . Since R/I embeds in a direct sum of copies of M , R/I has depth 1

and therefore is a one-dimensional CM ring. Also, $e(R/I) \leq 2$ since ideals of R/I are two-generated. Our claim now follows from the argument above, applied to the R/I -module M . To complete the proof, we show that R/I is isomorphic to an ideal of R . Taking duals over R , we note that $(R/I)^* \cong (0 : I)$, which, since $I \neq 0$, is an ideal of height 0 in R . Therefore $R/(0 : I)$ has positive multiplicity, and by [22, Chapter 3, Theorem 1.1] $(0 : I)$ is a principal ideal, that is, $(R/I)^*$ is cyclic. Choosing a surjection $R^* \twoheadrightarrow (R/I)^*$ and dualizing again, we have (since R/I is MCM and R is Gorenstein) $R/I \hookrightarrow R$ as desired. \square

2.2. Multiplicity at least four

Next we will show, in (2.5), that if $e(R) \geq 4$ then R has indecomposable MCM modules with arbitrarily large (constant) rank. In [9] Drozd and Roıter developed a machine for building big indecomposable modules over certain one-dimensional rings. Their approach was refined and generalized in [10,25]. The results in [25] apply to one-dimensional analytically unramified local rings and use the conductor square associated to the inclusion $R \hookrightarrow \bar{R}$, where \bar{R} is the integral closure of R in its total quotient ring. Here we observe that most of the theory goes through in the current setting.

As always, we assume that (R, \mathfrak{m}) is a CM local ring of dimension one with total quotient ring K . We let S be a finite birational extension of R ; that is, S is a subring of K containing R , and S is finitely generated as an R -module. Let \mathfrak{c} be the conductor of R in S , that is, the largest ideal of S that is contained in R .

We form the conductor square:

$$\begin{array}{ccc}
 R & \hookrightarrow & S \\
 \downarrow & & \downarrow \pi \\
 \frac{R}{\mathfrak{c}} & \hookrightarrow & \frac{S}{\mathfrak{c}}
 \end{array} \tag{1}$$

The bottom line of the square $\frac{R}{\mathfrak{c}} \hookrightarrow \frac{S}{\mathfrak{c}}$ is an Artinian pair in the terminology of [25]. (By definition, an Artinian pair is a module-finite extension of commutative Artinian rings.) A module over the Artinian pair $A \hookrightarrow B$ is a pair (V, W) where W is a finitely generated projective B -module, V is an A -submodule of W , and $BV = W$. Morphisms and direct sums are defined in the obvious way. We say that the $(A \hookrightarrow B)$ -module (V, W) has constant rank r provided W is a free R -module of rank r .

Lemma 2.2. *Suppose the Artinian pair $\frac{R}{\mathfrak{c}} \hookrightarrow \frac{S}{\mathfrak{c}}$ in (1) has an indecomposable module (V, W) of constant rank r . Then there is an indecomposable MCM R -module of constant rank r .*

Proof. Let P be a free S -module of constant rank r mapping onto W by change of rings. Define M by the pullback diagram

$$\begin{array}{ccc}
 M & \longrightarrow & P \\
 \downarrow & & \downarrow \pi \\
 V & \hookrightarrow & W
 \end{array} \tag{2}$$

As in [25] one checks that (2) is isomorphic to the pullback diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\subset} & SM \\
 \downarrow & & \downarrow \\
 \frac{M}{\mathfrak{c}M} & \xrightarrow{\subset} & \frac{SM}{\mathfrak{c}M}
 \end{array} \tag{3}$$

where SM is the S -submodule of $K \otimes_R M$ generated by the image of M , and the vertical arrows are the natural homomorphisms. It follows easily that any non-trivial decomposition of M would induce a decomposition (non-trivial by Nakayama’s lemma) of (V, W) . \square

We note that in the analytically unramified case the Drozd–Roïter conditions of Theorem 0.1 can be translated into conditions on the bottom line of the pullback diagram for $R \rightarrow \overline{R}$. The failure of these conditions is exactly what we need to build big indecomposables.

Theorem 2.3. *Let $A \hookrightarrow B$ be an Artinian pair, with (A, \mathfrak{m}, k) local. If either*

- (1) $v_A(B) \geq 4$, or
 - (2) $v_A(B) = 3$ and $\frac{\mathfrak{m}B}{\mathfrak{m}}$ is not cyclic as an A -module,
- then $A \hookrightarrow B$ has, for each n , an indecomposable module of constant rank n .

Proof. Although this result is not stated in the literature, it is proved in complete detail in Section 2 of [25]. The context is a bit different there, however, so a brief review of the proof is in order. In case (1), it is enough, by [25, (2.4)], to show that the Artinian pair $k \hookrightarrow B/\mathfrak{m}B$ has big indecomposables of large constant rank. The general construction in [25, (2.5)] (with $A := B/\mathfrak{m}B$) yields, by [25, (2.6)] and the discussion after its proof, the desired indecomposables except in the case where k is the 2-element field and B has at least 4 local components. In this case one can appeal to Dade’s theorem [7].

In case (2), we note that $\frac{\mathfrak{m}B}{\mathfrak{m}} \cong \frac{A+\mathfrak{m}B}{A}$ (since $\mathfrak{m}B \cap A = \mathfrak{m}$). We put $C = A + \mathfrak{m}B$ and $D = C/\mathfrak{m}C$. By [25, (2.4)] it is enough to show that the Artinian pair $k \hookrightarrow D$ has big indecomposables of constant rank. As shown in the proof of [25, (2.4)], $D \cong k[x, y]/(x^2, xy, y^2)$, and the existence of big indecomposables follows from case (i) of [25, (2.6)]. (The hypothesis that B is a principal ideal ring in [25, (2.3)] is irrelevant here, as it is used only in the case $v_A(B) = 2$.) \square

Corollary 2.4. *With R and S as above, suppose $v_R(S) \geq 4$. Then, for each $n \geq 0$, there is an indecomposable MCM R -module of constant rank n .*

Proof. Apply Theorem 2.3 and Lemma 2.2. \square

Finally we are ready to prove our first general result on unbounded CM type.

Theorem 2.5. *Let (R, \mathfrak{m}) be a one-dimensional CM local ring with $e(R) \geq 4$. Then R has, for each n , an indecomposable MCM module of constant rank n . In particular, R has unbounded CM type.*

By Corollary 2.4 it will suffice to prove the following:

Lemma 2.6. *Let (R, \mathfrak{m}, k) be a one-dimensional CM local ring with $e(R) = e$. Then R has a finite birational extension S with $v_R(S) = e$.*

Proof. Let $S_n = \text{End}_R(\mathfrak{m}^n) \subseteq K$, and put $S = \bigcup_n S_n$. To see that this works, we can harmlessly assume k is infinite. Let $Rf \subset \mathfrak{m}$ be a principal reduction of \mathfrak{m} . Choose n so large that

- (a) $\mathfrak{m}^{i+1} = f\mathfrak{m}^i$ for $i \geq n$ and
- (b) $v_R(\mathfrak{m}^i) = e$ for $i \geq n$.

Since f is a non-zero-divisor (as R is CM), it follows from (a) that $S = S_n$. We claim that $Sf^n = \mathfrak{m}^n$. We have $Sf^n = S_n f^n \subseteq \mathfrak{m}^n$. For the reverse inclusion, let $\alpha \in \mathfrak{m}^n$. Then $\frac{\alpha}{f^n} \mathfrak{m}^n \subseteq \frac{1}{f^n} \mathfrak{m}^{2n} = \frac{1}{f^n} (f^n \mathfrak{m}^n) = \mathfrak{m}^n$. Therefore $\frac{\alpha}{f^n} \in S$, and the claim follows. Now S is isomorphic to \mathfrak{m}^n as an R -module, and $v(S) = e$ by (b). \square

2.3. Multiplicity three

At this point we know that a one-dimensional CM local ring R has bounded CM type if $e(R) \leq 2$ and unbounded CM type if $e(R) \geq 4$. Now we address the troublesome case of multiplicity three for complete equicharacteristic hypersurfaces.

Let $R = k[[x, y]]/(f)$, where k is a field and $f \in (x, y)^3 - (x, y)^4$. If R is reduced, we know by (0.1) that R has bounded CM type if and only if R has finite CM type, that is, if and only if R satisfies the condition (dr2): $\frac{\mathfrak{m}R+R}{R}$ is cyclic as an R -module. If the characteristic is different from 2, 3, 5 there are simple normal forms [11] for f , classified by the Dynkin diagrams D_n, E_6, E_7, E_8 . (Of course the A_n singularities, of multiplicity two, have finite CM type too.) Normal forms have in fact been worked out in all characteristics [16,12], but the classification is complicated, particularly in characteristic 2. Here we focus on the case where R is not reduced.

Theorem 2.7. *Let $R = k[[x, y]]/(f)$, where k is a field and f is a non-zero non-unit of the formal power series ring $k[[x, y]]$. Assume*

- (1) $e(R) = 3$.
- (2) R is not reduced.
- (3) $R \not\cong k[[x, y]]/(xy^2)$.

For each positive integer n , R has an indecomposable MCM module of constant rank n .

The ring $k[[x, y]]/(xy^2)$ does indeed have bounded CM type; see the discussion following the proof and Theorem 2.8.

Proof. We know f has order 3 and that its factorization into irreducibles has a repeated factor. Thus, up to a unit, we have either $f = g^3$ or $f = g^2h$, where g and h are irreducible elements of $k[[x, y]]$ of order 1, and, in the second case, g and h are relatively prime. After a change of variables [29, Corollary 2, p. 137] we may assume that $g = y$.

In the second case, if the leading form of h is not a constant multiple of y , then by [29, Corollary 2, p. 137] we may assume that $h = x$. This is the case we have ruled out in (3).

Suppose now that the leading form of h is a constant multiple of y . By a corollary [29, Corollary 1, p. 145] of the Weierstrass Preparation Theorem, there exist a unit u and a non-unit power series $q \in k[[x]]$ such that $h = u(y + q)$. Moreover, $q \in x^2k[[x]]$ (since the leading form of h is a constant multiple of y). In summary, there are two cases to consider:

- (1) Case 1: $f = y^3$.
- (2) Case 2: $f = y^2(y + q)$, $0 \neq q \in x^2k[[x]]$.

Let \mathfrak{m} be the maximal ideal of R . We will show that R has a finite birational extension S such that $v_R(S) = 3$ and $\frac{\mathfrak{m}S}{\mathfrak{m}}$ is not cyclic as an R -module. An application of Lemma 2.2 and Theorem 2.3 will then complete the proof.

In Case (1) we put $S = R[\frac{y}{x^2}] = R + R\frac{y}{x^2} + R\frac{y^2}{x^4}$. Clearly $v_R(S) = 3$. It will suffice to show that $\frac{\mathfrak{m}S}{\mathfrak{m}^2S + \mathfrak{m}}$ is two-dimensional over R/\mathfrak{m} . We have

$$\mathfrak{m}S = \mathfrak{m} + R\frac{y}{x} + R\frac{y^2}{x^3} \quad \text{and} \quad \mathfrak{m}^2S + \mathfrak{m} = \mathfrak{m} + R\frac{y^2}{x^2}.$$

We must show (a) $\frac{y}{x} \notin \mathfrak{m} + R\frac{y^2}{x^2}$ and (b) $\frac{y^2}{x^3} \notin R\frac{y}{x} + \mathfrak{m} + R\frac{y^2}{x^2}$. If (a) fails, we multiply by x^2 and lift to $k[[x, y]]$, getting $xy \in (x^3, x^2y, y^2, y^3) \subseteq (x^2, y^2)$, contradiction. If (b) fails, we multiply by x^3 and lift to $k[[x, y]]$, getting $y^2 \in (x^2y, x^4, x^3y, xy^2, y^3) \subseteq (x, y)^3$, contradiction. This completes the proof in Case (1).

Assume now that we are in Case (2). Let \mathfrak{A}_r denote the set of rings $k[[x, y]]/y^2(y + q)$ with q an element of order r in $k[[x]]$. Then $R \in \mathfrak{A}_r$ for some $r \geq 2$. (The rings in \mathfrak{A}_1 are isomorphic to $k[[x, y]]/(xy^2)$, which we have ruled out.) Put $z = \frac{y}{x}$, and note that $z^3 + \frac{q}{x}z^2 = 0$. Therefore $R[z]$ is a finite birational extension of R and is isomorphic to a ring in \mathfrak{A}_{r-1} . If M is any indecomposable MCM $R[z]$ -module, then M is also a MCM R -module. Moreover, any R -endomorphism of M is also $R[z]$ -linear (as M is torsion-free and $R \rightarrow R[z]$ is birational). It follows that M is indecomposable as an R -module. Therefore the conclusion of the theorem passes from $R[z]$ to R , and we may assume that $R \in \mathfrak{A}_2$.

Thus $R = k[[x, y]]/y^2(y + q)$, where q is an element of order 2 in $k[[x]]$. Put $u := \frac{y}{x^2}$, $v := \frac{y^2 + qy}{x^5}$, and $S := R[u, v]$. The relations $u^2 = xv - \frac{q}{x^2}u$, $uv = v^2 = 0$ show that $S = R + Ru + Rv$, a finite birational extension of R . One checks easily that

$$\mathfrak{m}S = Rx + R\frac{y}{x} + R\frac{y^2 + qy}{x^4} \quad \text{and} \quad \mathfrak{m}^2S + \mathfrak{m} = Rx + Ry + R\frac{y^2 + qy}{x^3}.$$

To see that $v_R(S) = 3$, it suffices to show that $u \notin R + \mathfrak{m}S$ and $v \notin R + Ru + \mathfrak{m}S$. If $u \in R + \mathfrak{m}S (= R + R\frac{y}{x} + R\frac{y^2 + qy}{x^4})$, we would have (after multiplying by x^4 and lifting to $k[[x, y]]$)

$$x^2y \in (x^4, x^3y, y^2 + qy, y^2(y + q)). \tag{4}$$

In an equation demonstrating this inclusion, the coefficient of x^4 must be divisible by y . Cancelling y from such an equation and combining terms, we get $x^2 = Ax^3 + B(y+q)$, with $A, B \in k[[x, y]]$. Writing $q = Ux^2$ (where U is a unit of $k[[x]]$), we have $(1 - BU)x^2 = Ax^3 + By$. Since $x^2 \notin (x^3, y)$, B must be a unit of $k[[x, y]]$. But then $y \in (x^2)$, contradiction. Suppose now that $v \in R + Ru + mS$. Clearing denominators and lifting to $k[[x, y]]$, we get

$$y^2 + qy \in (x^5, x^3y, x(y^2 + qy), y(y^2 + qy)) \quad (5)$$

and it follows that $y^2 + qy \in (x^5, x^3y)$. Proceeding as before, we get an equation $y+q = Ax^3$. But then $(x^2) = (q) \subseteq (y, x^3)$, contradiction.

Finally, we show that $\frac{mS}{m^2S+m}$ is two-dimensional over R/m , that is, (a) $\frac{y}{x} \notin m^2S + m$ and (b) $\frac{y^2+qy}{x^4} \notin R\frac{y}{x} + m^2S + m$. If (a) were false, we could multiply by x^3 and get Eq. (4), which we have already seen to be impossible. Suppose now that (b) fails. Our worn-out argument yields (5) again, and the proof is complete. \square

The argument in the proof of Theorem 2.7 does not apply to the ring $R := k[[x, y]]/(xy^2 - y^3)$. Adjoining the idempotent $\frac{y^2}{x^2}$ to R , one obtains a ring isomorphic to $k[[x]] \times k[[x, y]]/(y^2)$, whose integral closure is $k[[x]] \times \bigcup_{n=1}^{\infty} R[\frac{y}{x^n}]$. From this information one can easily check that $\frac{mS}{m}$ is a cyclic R -module for every finite birational extension S of (R, m) , so we cannot apply Theorem 2.3. We appeal instead to the following result of Buchweitz et al.

Theorem 2.8 (Buchweitz et al. [3, Proposition 4.2]). *Let P be a two-dimensional regular local ring with maximal ideal m and let x, y be a generating set for m . Set $R := P/(xy^2)$. Then every indecomposable MCM R -module M has a presentation*

$$0 \longrightarrow P^n \xrightarrow{\varphi} P^n \longrightarrow M \longrightarrow 0$$

with $n = 1$ or $n = 2$ and φ one of the following matrices

$$\begin{aligned} n = 1 & : (y), (x), (y^2), (xy), (xy^2) \\ n = 2 & : \begin{pmatrix} y & x^k \\ 0 & -y \end{pmatrix}, \begin{pmatrix} xy & x^{k+1} \\ 0 & -xy \end{pmatrix}, \\ & \begin{pmatrix} xy & x^k \\ 0 & -y \end{pmatrix}, \begin{pmatrix} y & x^{k+1} \\ 0 & -xy \end{pmatrix}, \end{aligned}$$

where $k = 1, 2, 3, \dots$

Corollary 2.9. *Let P and R be as above. Then every indecomposable MCM R -module is generated by at most two elements.*

3. Summary

Let us summarize the results of the previous two sections:

Theorem 3.1. *Let k be any field, and let $R = k[[x_0, \dots, x_d]]/(f)$, where $d \geq 1$ and f is a non-zero non-unit of the power series ring $k[[x_0, \dots, x_d]]$.*

- (1) *Suppose $d = 1$ and R is reduced. Then R has bounded CM type if and only if R has finite CM type.*
- (2) *Suppose $d = 1$ and R is not reduced. Then R has infinite CM type. R has bounded CM type if and only if either $R \cong k[[x, y]]/(y^2)$ or $R \cong k[[x, y]]/(xy^2)$. In more detail:

 - (a) *If $R \cong k[[x, y]]/(y^2)$ or $R \cong k[[x, y]]/(xy^2)$, then every MCM R -module is generated by at most 2 elements.*
 - (b) *If $e(R) \geq 3$ and $R \not\cong k[[x, y]]/(xy^2)$, then R has, for each $n \geq 1$, an indecomposable MCM module of constant rank n .**
- (3) *Suppose $d \geq 2$ and the characteristic of k is different from 2. Then R has bounded CM type if and only if $R \cong k[[x_0, \dots, x_d]]/(g + x_2^2 + \dots + x_d^2)$ for some $g \in k[[x_0, x_1]]$ for which $k[[x_0, x_1]]/(g)$ is a one-dimensional ring of bounded CM type.*

Proof. Items (1) and (3) come from Theorem 0.1 and Proposition 1.5 respectively. In view of Theorems 2.1, 2.5 and 2.7 we know that, in the context of item (2), R has bounded CM type if and only if either $e(R) = 2$ or $R \cong k[[x, y]]/(xy^2)$. But if $e(R) = 2$ then (by an argument like that at the beginning of the proof of Theorem 2.7) one sees easily that $R \cong k[[x, y]]/(y^2)$. \square

Using this theorem and the analogous result for finite CM type [3,17], one can obtain higher dimensional examples of rings that have infinite bounded CM type. For example, take $g = x_1^2$ in (3) of Theorem 3.1.

We recall [27] that for one-dimensional local CM rings of finite CM type there is a universal bound on the multiplicities of the indecomposable MCM modules. In fact, if (R, \mathfrak{m}, k) is any one-dimensional local CM ring of finite CM type, then every indecomposable MCM R -module can be embedded in R^4 (in R^3 if k is algebraically closed). Since $e(R) \leq 3$, one obtains a bound of 12 on the multiplicities of the indecomposable MCM R -modules. The proof of Lemma 1.4 then gives a crude bound of 24 on the number of generators required for the indecomposable MCM modules. In fact, the sharp bound on the number of generators is probably about 8 and could be determined by a careful analysis of the work in [27]. It is interesting to observe that one can use Proposition 1.5 to get such universal bounds for higher-dimensional hypersurfaces. Here is a special case where the sharp bound in dimension one has been worked out:

Theorem 3.2. *Let k be an algebraically closed field of characteristic different from 2, 3, 5, and let $R \cong k[[x_0, \dots, x_d]]/(f)$, where f is a non-zero non-unit in the power series ring. If R has finite CM type, then every indecomposable MCM R -module can be generated by $6 \cdot 2^{d-1}$ elements.*

Proof. If $d = 1$, one can see from the computations in Chapter 9 of [28] that every indecomposable MCM R -module is generated by at most 6 elements. For $d > 1$ one uses the main theorem of [3,17] (the analog of Proposition 1.8 for finite CM type) to deduce that R is

obtained from a plane curve singularity of finite CM type by iterating the “sharp” operation $d - 1$ times. Then (2) of Proposition 1.5 provides the desired bound. \square

We have a corresponding result for hypersurfaces of bounded but infinite CM type. The proof is the same as that of Theorem 3.2. It is curious that the bound is better than in the case of finite type. The reason is that by item (2) of Theorem 3.1 the indecomposable MCM modules in dimension one are generated by two elements.

Theorem 3.3. *Let k be a field of characteristic not equal to 2, and let $R \cong k[[x_0, \dots, x_d]]/(f)$, where f is a non-zero non-unit in the power series ring. If R has bounded CM type but not finite CM type, then every indecomposable MCM R -module can be generated by 2^d elements.*

Note added in proof

In Theorems 1.1 and 3.1 (3), the form of the general hypersurface should be as in (†) of [30], if the field is not algebraically closed.

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