Theorem. Let \((S, n)\) be a Cohen-Macaulay local ring of dimension at least two, and let \(Z\) be an indeterminate. Then \(R := S[Z]/(Z^2)\) has unbounded Cohen-Macaulay type.

Proof. We will show that for every \(n \geq 2\) there is an indecomposable MCM \(R\)-module of rank \(n\). Fix \(n \geq 2\), and let \(W\) be a free \(S\)-module of rank \(2n\). Let \(I\) be the \(n \times n\) identity matrix and \(J\) the \(n \times n\) nilpotent Jordan block with \(t\) on the superdiagonal and \(s\) elsewhere. Let \(\{x, y\}\) be part of a minimal generating set for the maximal ideal \(m\) of \(S\), and put \(\varphi := xI + yJ\). Finally, put \(\psi := \begin{bmatrix} 0 & \varphi \\ 0 & 0 \end{bmatrix}\). Noting that \(\psi^2 = 0\), we make \(W\) into an \(R\)-module by letting \(z\) act as \(\psi\). Then \(W\) is a MCM \(R\)-module, and we claim that it is indecomposable.

Suppose \(W = U \oplus V\) as \(R\)-modules, with \(U \neq W\). We want to show that \(U = 0\). There is a \(2n \times 2n\) idempotent matrix \(\varepsilon\) such that \(U = \varepsilon(W) = \ker(1 - \varepsilon)\) and \(V = (1 - \varepsilon)(W) = \ker(\varepsilon)\).

Since \(U\) is an \(R\)-submodule of \(W\), we have \(\psi(U) \subseteq U\), that is, \((1 - \varepsilon)\psi \varepsilon = 0\). Similarly, since \(\psi(V) \subset V\), we have \(\varepsilon\psi(1 - \varepsilon) = 0\). Combining these two equations, we have

\[
\psi \varepsilon = \varepsilon \psi. \tag{1}
\]

Write \(\varepsilon = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\), where each block is \(n \times n\). From (1), we obtain the equation

\[
\begin{bmatrix}
\gamma x + J \gamma y & \delta x + J \delta y \\
0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & \alpha x + \alpha J y \\
0 & \gamma x + \gamma J y
\end{bmatrix}. \tag{2}
\]

Since \(x + n^2\) and \(y + n^2\) are linearly independent over \(k := S/n\), we get the equations

\[
\bar{\gamma} = 0, \quad \bar{\delta} = \bar{\alpha}, \quad \bar{J} \bar{\delta} = \bar{\alpha} \bar{J}, \tag{3}
\]

where the bars denote reductions modulo \(n\). Therefore \(\bar{\varepsilon} = \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ 0 & \bar{\alpha} \end{bmatrix}\), and \(\bar{\alpha} \bar{J} = \bar{J} \bar{\alpha}\). Since \(\bar{\alpha}\) commutes with the non-derogatory matrix \(\bar{J}\), \(\bar{\alpha}\) belongs to \(k[J]\). In particular, \(\bar{\alpha}\) is upper-triangular with a constant, say \(a\), on the diagonal.

Since \(U \neq W\), Nakayama’s lemma implies that \(\bar{\varepsilon}\) is not surjective, whence \(a = 0\). Therefore \(\bar{\varepsilon}^{2n} = 0\), and, since \(\bar{\varepsilon}^2 = \bar{\varepsilon}\), we have \(\bar{\varepsilon} = 0\). By Nakayama’s lemma, \(1 - \varepsilon\) is surjective and, being idempotent, must be equal to the identity matrix. Thus \(U = 0\), as desired. \(\square\)

References