Bounded and Finite Representation Type, an introduction

Say: all rings commutative, all modules f.g., most rings Noetherian.
Let's look around for other structure theorems for modules over nice rings.

Proposition 1. *A finitely generated abelian group decomposes as a direct sum of a free module of finite rank ($\mathbb{Z}^r$) and cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$.*

In fact, since $\mathbb{Z}$ is a PID and $\mathbb{Z}$-modules are exactly the abelian groups, this is a special case of the following structure theorem:

Theorem 2. *Let R be a principal ideal domain (e.g., $\mathbb{Z}$ or $k[x]$). Then the only indecomposable finitely generated R-module is a direct sum of a free module of finite rank and cyclic modules $R/(f^n)$, f a prime element.*

What I want to talk about today is the problem of finding **converses** to this sort of theorem. For example:

Proposition 3. *Let R be a Noetherian ring such that the only indecomposable torsion-free module is the free module R. Then R is a PID.*

For another example:

Questions. ("FGC ring", Kaplansky 1950s) For what rings is every finitely generated module a direct sum of cyclic modules?

Answer: for Noetherian rings, just the PIDs. For non-Noetherian rings, much more complicated:

- f.g. ideals are principal
- each ideal has fin.many minimal primes
- if I has a unique minimal prime P, the ideals from I to P form a chain
- $R_{\mathfrak{m}}$ is an almost maximal valuation ring (for every proper homomorphic image, every set of pairwise solvable congruences modulo ideals I_α is simultaneously solvable).

From now on, we stick to Noetherian rings.
We know what happens if every ideal is principal. What if every ideal is at most 2-generated?

Theorem 4 (Bass '63). *Let R be a one-dimensional Noetherian ring in which every ideal is two-generated. Then every torsion-free R-module is isomorphic to a direct sum of ideals.*

Example 5. Some rings in which every ideal is 2-generated:

- $k[x, y]/(f)$ for any polynomial f of degree 2
- rings of integers in quadratic number fields (e.g., $\mathbb{Z}[\sqrt{-17}]$).

Well, don't stop now! What about rings in which every ideal is 3-generated?! Bad news: $k[[t^3, t^7, t^8]]$ is such a ring, and has infinitely many indecomposable torsionfree modules. In fact, it has indecomposable torsionfree modules of unbounded rank.

How do I know?

First, a little **Notation and Definitions:**

- $(R, \mathfrak{m}, k)$ is a one-dimensional local ring containing a nonzerodivisor
- K is the total quotient ring (invert all the nonzerodivisors).
- $\widetilde{R}$ is the integral closure in K

Theorem 6. *(Drozd-Roĭter '67, Green-Reiner '78) TFAE:*

(1) *R has only finitely many indecomposable torsionfree modules, up to isomorphism (**FRT**).*
(2) *There is a bound on the multiplicities of indecomposble torsionfree R-modules, and $\widetilde{R}$ is a finitely generated R-module (**BRT**).*
(3) *R satisfies*
(DR1) $\mu_R(\widetilde{R}) \leq 3$
(DR2) $\mathfrak{m}\widetilde{R}/\mathfrak{m}$ *is a cyclic R-module.*

Note: So, for example, you can write down all the complete domains of FRT that contain $\mathbb{C}$:

$$k[[t^2, t^n]], k[[t^3, t^4]], k[[t^3, t^5]], k[[t^3, t^4, t^5]], k[[t^3, t^5, t^7]]$$

Note: That business about $\widetilde{R}$ being finitely generated is to rule out cases like $k[[x, y]]/(y^2)$, over which all the elements y/x^n are integral, so its integral closure is not a finitely generated module.

What if you remove that assumption in (2), but still assume there's a bound?

Theorem 7 (L–Wiegand (UNL))**.** *Let k be an infinite field. Then the following is a complete list, up to isomorphism, of the complete one-dimensional CM local rings R with bounded but infinite representation type:*

(1) $A := k[[x, y]]/(y^2)$
(2) $D := k[[x, y]]/(xy^2)$
(3) $T := \mathrm{End}_T(\mathfrak{m}_T) \cong k[[x, y, z]]/(xy, yz, z^2)$

Higher dimensions: Now drop the assumption that R have dimension one.

Assume, though, that R is *Cohen–Macaulay*. This means that there is a nonzerodivisor x so that $R/(x)$ is Cohen–Macaulay, and all zero-dimensional rings are CM. Geometrically, this is like an *unmixedness* condition: locally, the components of the variety corresponding to R all have the same dimension, and that still holds true when you take a hyperplane section.

The modules we want to consider now are the *maximal Cohen–Macaulay* ones. This means that there's a nzd x so that M/xM is MCM over $R/(x)$. This is the appropriate generalization of torsionfree, to avoid the embarrassment of always having infinitely many modules lying around.

So now

Definition 8. Say R has *finite Cohen–Macaulay type* or *FCMT* if there are only finitely many indecomposable MCM R-modules up to isomorphism.

Note: We saw that fields k, PIDs, have FCMT. In fact, all *regular rings* have FCMT. We also saw that we know everything there is to know about zero-dimensional and one-dimensional rings (thanks to DR).

What about dimension two? Note that in this case, MCM is equivalent to reflexive: so the natural map $M \to M^{**}$ taking m to "evaluation at m" is a isomorphism.

Theorem 9 (Herzog '78)**.** *Let $S = k[[x,y]]$ be a power series ring and let G be a finite subgroup of $GL(2,k)$ with $char\ k \nmid |G|$, acting linearly on S. Let $R = S^G$. Then R has only finitely many indecomposable reflexive modules up to isomorphism.*

Theorem 10 (Auslander–Reiten '82)**.** *Rings obtained in this way are the only complete local rings of FCMT in dimension two.*

So that's kind of the whole story there. There is no such theorem in dimension three yet, as we'll see in a moment.

Another place we know the whole story: **Hypersurfaces**.

Theorem 11. *The following is a complete list, up to isomorphism, of hypersurfaces $k[[x, y, z_1, \ldots, z_n]]/(f)$ with FCMT (need char $k > 5$.)*

$$
\begin{array}{rl}
(A_n) & f = x^{n+1} + y^2 + z_1^2 + \cdots + z_n^2 \\
(D_n) & f = x^{n-1} + xy^2 + z_1^2 + \cdots + z_n^2 \\
(E_6) & f = x^3 + y^4 + z_1^2 + \cdots + z_n^2 \\
(E_7) & f = x^3 + xy^3 + z_1^2 + \cdots + z_n^2 \\
(E_8) & f = x^3 + y^5 + z_1^2 + \cdots + z_n^2
\end{array}
$$

(Note that these are the so-called "simple" hypersurfaces according to Ar'nold's classification.)

Other than that (i.e., dim ≤ 2 and hypersurfaces), only two examples are known of rings of FCMT. So we have to look for other questions.

Theorem 12 (Auslander '87)**.** *Let R be a complete CM local ring of FCMT. Then R has at most an isolated singularity.*

Huneke and I reproved this in '00 by completely different methods and removed the assumption "complete".

But that raises a question: Does FCMT ascend to and descend from the completion?

Answer: Yes (L–Wiegand, '98) under a very mild technical condition.

Possible things to **natter on about:**

- the homogeneous (geometric) case
- countable CM type
- ???
- profit!!!