Factoring the Adjoint and Maximal CM Modules

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(joint work with Ragnar-Olaf Buchweitz)

Let k be a field and $X = (x_{ij})$ the generic $(n \times n)$–matrix over k. Put $S = k[x_{ij}]$. Let $\mathrm{adj}(X)$ denote the "classical adjoint" of X, whose entries are the appropriately signed submaximal minors (or cofactors) of X, and which is characterized by the matrix equation

$$X \ \mathrm{adj}(X) = \det(X) \cdot \mathrm{id}_n = \mathrm{adj}(X) \ X \,. \tag{1}$$

Our motivating question is due to G.M. Bergman [1], who asked whether the equation (1), viewed as a factorization of the diagonal matrix $\det(X) \cdot \mathrm{id}_n$, can be refined by writing $\mathrm{adj}(X) = YZ$ for a pair of noninvertible $(n \times n)$–matrices Y and Z over S. He gave a partial answer to the question:

Theorem (Bergman)**.** *Let k be an algebraically closed field of characteristic zero.*

(a) *For n odd, there are no nontrivial factorizations of* $\mathrm{adj}(X)$*.*
(b) *For n even, any factorization* $\mathrm{adj}(X) = YZ$ *must have either* $\det Y = \det X$ *or* $\det Z = \det X$*, up to units of S.*

We translate Bergman's question into commutative algebraic terms, as follows: The pair $(X, \mathrm{adj}(X))$ forms a *matrix factorization* of $\det X \in S$, in the sense of Eisenbud [3]. In particular, $M := \mathrm{cok}\,\mathrm{adj}(X)$ and $L := \mathrm{cok}\, X$ are *maximal Cohen–Macaulay (MCM) modules* over the hypersurface $R := S/(\det X)$. The existence of a nontrivial factorization $\mathrm{adj}(X) = YZ$ is equivalent to a nonsplit short exact sequence

$$0 \longrightarrow \mathrm{cok}\, Z \longrightarrow M \longrightarrow \mathrm{cok}\, Y \longrightarrow 0\,,$$

of maximal Cohen–Macaulay modules over R. The MCM R-modules are not particularly well understood, but it follows from Bruns' calculation of the divisor class group [2] that the only MCM R-modules of rank one, up to isomorphism, are $L = \mathrm{cok}\, X$ and the dual $L^\vee := \mathrm{cok}\, X^T$. This translation already allows us to recover Bergman's result for $n = 3$ and any UFD coefficient ring k:

Proposition. *Let $X = (x_{ij})$ be the generic (3×3)–matrix over a unique factorization domain k. Then there are no nontrivial factorizations of* $\mathrm{adj}(X)$ *over* $k[x_{ij}]$*.*

For $n \geq 4$, we consider the case $\det Y = u \det X$, u a unit in S. This corresponds precisely to assuming that either $\mathrm{cok}\, Y \cong L$ or $\mathrm{cok}\, Y \cong L^\vee$. A pushout construction reduces the open case of Bergman's result to the problem of classifying all extensions

$$0 \longrightarrow \mathrm{cok}\, Y \longrightarrow Q \longrightarrow L \longrightarrow 0\,,$$

where either $\mathrm{cok}\, Y \cong L$ or $\mathrm{cok}\, Y \cong L^\vee$. In other words, we must compute $\mathrm{Ext}^1_R(L, L)$ and $\mathrm{Ext}^1_R(L, L^\vee)$. The first case follows from a recent result of R. Ile [5]:

Theorem (Ile)**.** $\operatorname{Ext}^1_R(L,L)=0$.

On the other hand, computer calculations [4] reveal that $\operatorname{Ext}^1_R(L,L^\vee)\neq 0$. To better understand the structure of $\operatorname{Ext}^1_R(L,L^\vee)$, we consider first $\operatorname{Hom}_R(M,L^\vee)$.

Theorem. *The R-module $\operatorname{Hom}_R(M,L^\vee)$ is maximal Cohen–Macaulay of rank $n-1$, generated by $\binom{n}{2}$ elements. Indeed,* $\operatorname{Hom}_R(M,L^\vee)$ *is generated by the* alternating *matrices over S. More precisely, for any alternating $(n\times n)$–matrix with entries in S, there exists a unique alternating matrix B_A of the same size such that*

$$A\operatorname{adj}(X)=X^TB_A\,,$$

and $\operatorname{Hom}_R(M,L^\vee)$ *consists of all homomorphisms induced by such pairs (A,B_A). The entries of $B_A=(b_{ij})$ are given in terms of those of $A=(a_{kl})$:*

$$b_{ij}=\sum_{k<l}(-1)^{i+j+k+l}a_{kl}[ij\widehat{\,|\,}kl]\,,$$

where $[ij\widehat{\,|\,}kl]$ denotes the $(n-2)\times(n-2)$ minor of X obtained by removing the i,j rows and k,l columns.

In particular, we obtain an answer to the open case of Bergman's question:

Theorem. *When n is even, there exist* invertible *alternating matrices A over S; for such A, we have* $\operatorname{adj}(X)=(A^{-1}X^T)B_A$*, a nontrivial factorization of the adjoint.*

Returning to $\operatorname{Ext}^1_R(L,L^\vee)$, we compute the minimal graded S-free resolution and obtain

Theorem. $\operatorname{Ext}^1_R(L,L^\vee)$ *is a MCM module of rank one over $S/I_{n-1}(X)$, the ring defined by the submaximal minors of X. For each nonzero alternating matrix A with polynomial entries, there is an extension of $L^\vee$ by L*

$$0\longrightarrow L^\vee\longrightarrow Q\longrightarrow L\longrightarrow 0\,,$$

with Q an orientable MCM R-module of rank 2, given by the matrix factorization

$$\left(\begin{pmatrix}X^T & A\\ 0 & X\end{pmatrix},\begin{pmatrix}\operatorname{adj}(X)^T & -B\\ 0 & \operatorname{adj}(X)\end{pmatrix}\right).$$

Considering the middle terms Q of extensions in $\operatorname{Ext}^1_R(L,L^\vee)$, we observe that for $n\geq 3$, the MCM-representation theory of the generic determinantal hypersurface is quite "wild", even restricted to orientable MCM modules of rank 2.

Theorem. *Assume $n\geq 3$. Then there is a surjection from the isomorphism classes of extensions $L^\vee$ by L to the principal ideals of a polynomial ring over k in $(n-2)^2$ variables. In particular, the MCM R-modules of rank 2 cannot be parametrized by the points of any finite-dimensional variety over k.*

This last result stands in stark contrast to the situation when $n=2$, wherein there are only three indecomposable MCM modules up to isomorphism: R, L, and $L^\vee$.

References

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