

Semidualizing Modules and Gorenstein Presentations

joint work with D.A. Jorgensen and S. Sather-Wagstaff

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Definition of dualizing module

Standing assumptions throughout: (R, \mathfrak{m}, k) is a Cohen–Macaulay local ring, and all modules are finitely generated.

Definition (Grothendieck '57)

Say that an R -module ω is **dualizing** if

- ▶ the natural map

$$R \longrightarrow \mathrm{Hom}_R(\omega, \omega),$$

sending r to “multiplication by r ,” is an isomorphism;

- ▶ $\mathrm{Ext}_R^i(\omega, \omega) = 0$ for all $i > 0$; and
- ▶ ω has finite injective dimension.

Examples

Example

If R is Artinian, then $E_R(k)$, the injective hull of the residue field, is dualizing.

Example

The free module R is dualizing if and only if R is Gorenstein. (This can be taken as the definition of Gorensteinity.)

Example

There exist Cohen–Macaulay local rings R having no dualizing module [Ferrand-Raynaud '70]. (Precisely: there is a 1-dimensional local domain R such that the completion \widehat{R} is not generically Gorenstein.)

Further properties

- ▶ If a dualizing module ω exists, then it is unique up to (non-canonical) isomorphism.
- ▶ Whence the name “dualizing”:

Theorem

Let M be a maximal Cohen–Macaulay R -module. Then M is *totally ω -reflexive*, meaning

- ▶ $\text{Ext}_R^i(M, \omega) = 0$ for all $i > 0$,
- ▶ $\text{Ext}_R^i(\text{Hom}_R(M, \omega), \omega) = 0$ for all $i > 0$, and
- ▶ the natural map

$$M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, \omega), \omega)$$

is an isomorphism.

Thus the assignment $M \rightsquigarrow M^\vee = \text{Hom}_R(M, \omega)$ is a duality on the MCM R -modules.

Existence of dualizing modules

Theorem (Sharp '71)

Suppose that R is a homomorphic image of a Gorenstein local ring Q . Then R has a dualizing module.

Sketch of proof.

Cut down by regular sequences to assume that Q and R are Artinian. Then $\text{Ext}_Q^i(R, Q) = 0$ for all $i > 0$, and $\text{Hom}_Q(R, Q)$ is dualizing for R . \square

Consequences

- ▶ If R is complete — so in particular if R is Artinian — then R has a dualizing module by the Cohen Structure Theorem.
- ▶ If R is essentially of finite type over a ring with a dualizing module, then R has a dualizing module.

A converse to Sharp's theorem

Theorem (Foxby '72, Reiten '72)

If R has a dualizing module ω , then R is a homomorphic image of a Gorenstein local ring Q .

Sketch of proof.

Let $Q = R \ltimes \omega$ be Nagata's idealization of ω , so:

- ▶ as R -modules, $R \ltimes \omega = R \oplus \omega$, and
- ▶ multiplication is given by

$$(r, c)(r', c') = (rr', rc' + r'c).$$

With this structure, $R \ltimes \omega$ is a commutative local ring with maximal ideal $\mathfrak{m} \oplus \omega$, and surjects onto R via $(r, c) \mapsto r$. The ideal $0 \oplus \omega$ is square-zero.

From the fact that ω is R -dualizing, one checks that $R \ltimes \omega$ is Gorenstein. □

Definition of semidualizing module

Definition (Foxby '72, Vasconcelos '74, Golod '84, Wakamatsu '88)

Say that an R -module C is **semidualizing** (a.k.a. (PG), spherical, $G \dots$) if

- ▶ the natural map $R \longrightarrow \text{Hom}_R(C, C)$, sending r to “multiplication by r ,” is an isomorphism; and
- ▶ $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Remarks

- ▶ A semidualizing module C is dualizing if and only if it has finite injective dimension.
- ▶ The free module R is always semidualizing.
- ▶ If R is Gorenstein, then the only semidualizing module is R . The converse holds if R has a dualizing module.

Further properties

- ▶ A semidualizing module C gives a duality only on the **totally C -reflexive** modules M , that is, the ones such that
 - ▶ $\text{Ext}_R^i(M, C) = 0$ for all $i > 0$,
 - ▶ $\text{Ext}_R^i(\text{Hom}_R(M, C), C) = 0$ for all $i > 0$, and
 - ▶ $M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism.
- ▶ Total reflexivity gives an order-type relation on the set of semidualizing modules: $C \trianglerighteq D$ if C is totally D -reflexive. Under this relation R is the maximum element, and a dualizing module ω is the minimum. It is not known whether this relation is transitive.
- ▶ If C is semidualizing and R has a dualizing module ω , then the dual $C^\vee = \text{Hom}_R(C, \omega)$ is again a semidualizing module. If $C \not\cong \omega, R$, then C^\vee is never isomorphic to C .

Examples of non-trivial semidualizing modules

Example (Foxby '87)

Let A be a complete local ring which is Cohen–Macaulay but not Gorenstein, and set

$$R = A[[x, y]]/(x^2, xy, y^2).$$

Then R is again complete, CM, and non-Gorenstein, so has a dualizing module ω which is not free.

Set $C = \text{Hom}_A(R, A)$. Then C is a semidualizing R -module, and $R \not\cong C \not\cong \omega$. The same is true of $C^\vee = \text{Hom}_R(C, \omega)$.

Note that R has a non-trivial decomposition as a tensor product. Indeed, we can write $A = Q/J$ for some Gorenstein Q , and then

$$R \cong \frac{Q[[x, y]]}{JQ[[x, y]]} \otimes_{Q[[x, y]]} \frac{Q[[x, y]]}{(x, y)^2}.$$

Existence of semidualizing modules

Theorem (DJ-GL-SSW '08)

Suppose that R has a dualizing module ω and a non-trivial semidualizing module C . Then there exists a Gorenstein local ring Q with ideals I_1, I_2 such that

- ▶ $R = Q/(I_1 + I_2) = Q/I_1 \otimes_Q Q/I_2$;
- ▶ $\mathrm{Tor}_i^Q(Q/I_1, Q/I_2) = 0$ for all $i > 0$;
- ▶ each Q/I_j is Cohen–Macaulay and not Gorenstein; and
- ▶ R is totally reflexive over each Q/I_j .

Proof of main theorem

$$\begin{array}{ccc} R_1 = R \ltimes C & & R_2 = R \ltimes C^\vee \\ & \searrow & \swarrow \\ & R & \end{array}$$

Let R_1 be the idealization of C , and R_2 the idealization of C^\vee .

Then R_1 and R_2 are CM but not Gorenstein (as C , C^\vee are not dualizing).

The dualizing modules for R_1 and R_2 are

$$D_1 = \text{Hom}_R(R_1, \omega) \quad \text{and} \quad D_2 = \text{Hom}_R(R_2, \omega).$$

Proof cont'd

$$\begin{array}{ccc} & Q = R_1 \times D_1 = R_2 \times D_2 & \\ & \swarrow \qquad \searrow & \\ R_1 = R \times C & & R_2 = R \times C^\vee \\ & \searrow \qquad \swarrow & \\ & R & \end{array}$$

The idealization of the R_1 -module D_1 is a Gorenstein ring Q , which is isomorphic as an R -module to

$$Q \cong R \oplus C \oplus C^\vee \oplus \omega.$$

We show that Q is isomorphic (as a ring!) to $R_2 \times D_2$.

Proof cont'd

$$\begin{array}{ccc} & Q = R_1 \times D_1 = R_2 \times D_2 & \\ & \swarrow \qquad \searrow & \\ R_1 = R \times C & & R_2 = R \times C^\vee \\ & \searrow \qquad \swarrow & \\ & R & \end{array}$$

To show $\mathrm{Tor}_i^Q(R_1, R_2) = 0$ for all $i > 0$, we work our way around the diamond, showing first that

$$\mathrm{Tor}_i^R(R_1, R_2) = 0 \quad \text{then} \quad \mathrm{Tor}_i^{R_1}(Q, R) = 0,$$

and finally the desired vanishing.