ASCENT OF FINITE COHEN-MACAULAY TYPE

Graham Leuschke and Roger Wiegand

17 December 99

In 1987 F.-O. Schreyer conjectured that a local ring R has finite Cohen-Macaulay type if and only if the completion $\widehat{R}$ has finite Cohen-Macaulay type. We prove the conjecture for excellent Cohen-Macaulay local rings and also show by example that it can fail in general.

Introduction

Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of dimension d. Recall that a non-zero R-module M is *maximal Cohen-Macaulay* (MCM) provided it is finitely generated and there exists an M-regular sequence $(x_1, \dots, x_d)$ in the maximal ideal $\mathfrak{m}$. The ring R is said to have *finite Cohen-Macaulay type* (or finite CM type) if there are, up to isomorphism, only finitely many indecomposable MCM R-modules.

There has been a great deal of progress in recent years on the problem of classifying all local rings of finite CM type. Work on this problem has proceeded in two main directions. First, the complete equicharacteristic hypersurface singularities of finite CM type have been completely characterized ([GK], [GKr], [BGS], [K], [So]). The other approach has been to reduce to the complete case, where one can apply a beautiful theorem of Auslander [A]: If a complete CM local ring R has finite CM type, then R is an isolated singularity. In this latter direction, the main focus has been on a conjecture of Schreyer [Sc], which states that a local ring R has finite CM type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite CM type. In [W], the second-named author proved that finite CM type satisfies faithfully flat descent, provided the closed fiber is Cohen-Macaulay, which establishes one direction of the conjecture. In this paper we prove the other direction. Specifically, we prove the following theorem.

Main Theorem. *Let $(R, \mathfrak{m})$ be an excellent CM local ring. Then R has finite Cohen-Macaulay type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite Cohen-Macaulay type.*

As a corollary we deduce that Auslander's theorem holds for all excellent CM local rings.

The assumption that R be excellent, which is absent from Schreyer's original conjecture, is used in two different ways in our proof. It seems unlikely to us that it could be eliminated.

1991 *Mathematics Subject Classification.* Primary: 13C14, 13H10, 13B40; Secondary: 13D02, 13C05.
Key words and phrases. finite Cohen-Macaulay type, canonical module.
The research of the second-named author was partially supported by the National Science Foundation.

In the second part of the paper, we address the analogous problem for non-Cohen-Macaulay local rings. We give two examples to show that there exist local rings with *no* MCM modules, whose completions have infinite CM type.

Throughout, all rings will be commutative and Noetherian, and all modules will be finitely generated. We abbreviate the assertion "M is isomorphic to a direct summand of N" to "$M \mid N$". Denote by $\operatorname{syz}_R^n(M)$ the n^{th} syzygy in an arbitrary free resolution of an R-module M; it is unique up free direct summands. To say that a property holds on the punctured spectrum of a local ring $(R, \mathfrak{m})$ means that the property holds for all localizations $R_\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$. Finally, we write R^{h} and $\widehat{R}$ for the Henselization and completion, respectively, of a local ring R.

Proof of the Main Theorem

The second-named author [W] established descent of finite CM type along flat local homomorphisms with CM closed fiber.

Theorem 1.1 ([W, 1.4])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism. Let $\mathcal{C}(R)$ and $\mathcal{C}(S)$ be classes of finitely generated R-modules (resp. S-modules) which are closed under taking direct summands, and assume that $S \otimes_R M \in \mathcal{C}(S)$ whenever $M \in \mathcal{C}(R)$. If $\mathcal{C}(S)$ contains only finitely many indecomposable modules up to isomorphism, the same holds for $\mathcal{C}(R)$.*

Accordingly, we focus on the question of ascent of finite CM type. We will rely on the following theorem, which reduces the problem to proving that the localization $R_\mathfrak{p}$ is Gorenstein for all non-maximal primes $\mathfrak{p}$.

Theorem 1.2 ([W, 2.9])**.** *Let $(R, \mathfrak{m})$ be an excellent CM local ring which is Gorenstein on the punctured spectrum. If R has finite CM type, then $\widehat{R}$ has finite CM type.*

The property that we will use to verify that a ring is Gorenstein on the punctured spectrum involves the canonical module ω_R. See [BH, Chapter 3] for basic properties of the canonical module. We also record the following lemma from [EG]; note that this is not the precise statement that appears there, but is what is actually proved.

Lemma 1.3 ([EG, 3.8])**.** *Let $(R, \mathfrak{m})$ be a local ring satisfying (S_k) which is Gorenstein in codimension $k - 1$. Then every R-module satisfying (S_k) is a k^{th} syzygy of some finitely generated R-module.*

Lemma 1.4. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Then R is Gorenstein on the punctured spectrum if and only if ω is a direct summand of a d^{th} syzygy of some finitely generated R-module.*

Proof. If R is Gorenstein on the punctured spectrum, then every MCM R-module, in particular ω, is a d^{th} syzygy by Lemma 1.3. For the converse, let M be a finitely generated R-module so that $\omega \mid \operatorname{syz}_R^d(M)$. We have an exact sequence

$$0 \to \operatorname{syz}_R^d(M) \to F \to \operatorname{syz}_R^{d-1}(M) \to 0,$$

with F a finitely generated free R-module. Use the split surjection $\operatorname{syz}_R^d(M) \twoheadrightarrow \omega$ to form a pushout diagram

$$(\dagger) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{syz}_R^d(M) & \longrightarrow & F & \longrightarrow & \operatorname{syz}_R^{d-1}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & \omega & \longrightarrow & X & \longrightarrow & \operatorname{syz}_R^{d-1}(M) & \longrightarrow & 0. \end{array}$$

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a nonmaximal prime. Since $\omega_\mathfrak{p}$ is a canonical module for $R_\mathfrak{p}$ and $\operatorname{syz}_R^{d-1}(M)_\mathfrak{p}$ is a MCM $R_\mathfrak{p}$-module, the second row of (†) splits when localized at $\mathfrak{p}$. It is a straightforward diagram chase to show that $\omega_\mathfrak{p}$ is a direct summand of $F_\mathfrak{p}$, and so is free. Hence $R_\mathfrak{p}$ is Gorenstein, as desired. □

Not every CM local ring has a canonical module (see, for example, [RWW]). Thus, in order to apply the lemma, we change base to a ring which is known to have a canonical module. Let R be an excellent CM local ring. Then the Henselization R^{h} of R is also excellent, by [G, 5.3]. It is a consequence of Néron-Popescu desingularization [Sw, 2.4] that an excellent Henselian local ring satisfies the Artin approximation property. Hinich showed in [Hi] that Artin approximation implies the existence of a canonical module (see also [R]).

For a local ring R of dimension d, we let $\mathcal{C}(R)$ be the class of all R-modules M such that M is isomorphic to a direct summand of a d^{th} syzygy of a finitely generated R-module. Note that, since we do not require free resolutions to be minimal, $R \in \mathcal{C}(R)$.

Definition 1.5. We say that a local ring $(R, \mathfrak{m})$ of dimension d has *finite syzygy type* provided there are, up to isomorphism, only finitely many indecomposable modules in $\mathcal{C}(R)$.

It is clear that finite CM type implies finite syzygy type for CM local rings. See Corollary 1.8 for a partial converse. Just as important for our purposes, finite syzygy type ascends to the Henselization.

Proposition 1.6. *Let $(R, \mathfrak{m})$ be a local ring with Henselization R^{h}. Then R has finite syzygy type if and only if R^{h} has finite syzygy type.*

Proof. Descent follows from the general result of Wiegand reproduced above as Theorem 1.1, since $R \to R^{\mathrm{h}}$ is a flat local homomorphism. For ascent, it suffices by [W, 2.1] to show that if an R^{h}-module M is a direct summand of a d^{th} syzygy over R^{h}, then M is a direct summand of $R^{\mathrm{h}} \otimes_R N$ for some d^{th} syzygy ${}_RN$.

Define $\mu : R^{\mathrm{h}} \otimes_R R^{\mathrm{h}} \to R$ by $\mu(a \otimes b) = ab$, and let J be the kernel of μ. We claim that the exact sequence

$$(*) \qquad 0 \longrightarrow J \longrightarrow R^{\mathrm{h}} \otimes_R R^{\mathrm{h}} \xrightarrow{\ \mu\ } R^{\mathrm{h}} \longrightarrow 0$$

splits as $R^{\mathrm{h}} \otimes_R R^{\mathrm{h}}$-modules. This property is referred to as *separability* in [DI]. By [DI, 7.1], the extension $R \to R^{\mathrm{h}}$ is separable if and only if $R/\mathfrak{m} \to R^{\mathrm{h}}/\mathfrak{m}R^{\mathrm{h}}$ is separable. But $\mathfrak{m}R^{\mathrm{h}}$ is the maximal ideal of R^{h}, and $R^{\mathrm{h}}/\mathfrak{m}R^{\mathrm{h}} = R/\mathfrak{m}$. This extension is clearly separable, so $R \to R^{\mathrm{h}}$ is separable as well. This proves the claim.

Let X be a finitely generated R^{h}-module such that $M \mid \operatorname{syz}^d_{R^{\mathrm{h}}}(X)$. Applying $- \otimes_{R^{\mathrm{h}}} X$ to the sequence $(*)$, we get a split exact sequence of R^{h}-modules

$$0 \longrightarrow J \otimes_{R^{\mathrm{h}}} X \longrightarrow R^{\mathrm{h}} \otimes_R X \longrightarrow X \longrightarrow 0.$$

Thus X is a direct summand of the extended module $R^{\mathrm{h}} \otimes_R X$, where the action of R^{h} on $R^{\mathrm{h}} \otimes_R X$ is via change of rings. Write ${}_RX$ as a directed union of finitely generated R-modules Y_α. Then, since X is finitely generated as an R^{h}-module, $X \mid R^{\mathrm{h}} \otimes_R Y_\alpha$ for some Y_α. Set $Z = \operatorname{syz}^d_R(Y_\alpha)$. Then there exists a free module $(R^{\mathrm{h}})^n$ so that $M \mid (R^{\mathrm{h}} \otimes_R Z) \oplus (R^{\mathrm{h}})^n$ as R^{h}-modules. Put $N = Z \oplus R^n$ to finish the proof. □

Proposition 1.7. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Let $\underline{x} = \{x_1, \dots, x_d\}$ be a system of parameters for R, and let $\underline{x}^n = \{x_1^n, \dots, x_d^n\}$. For each integer $n \geq 1$, let Σ_n be the set of isomorphism classes of R-modules appearing in direct-sum decompositions of direct sums of copies of $\operatorname{syz}^d_R(\omega/(\underline{x}^n)\omega)$. Set $\Sigma := \bigcup_n \Sigma_n$. If Σ contains only finitely many isomorphism classes of indecomposables, then R is Gorenstein on the punctured spectrum.*

Proof. Fix an integer $n \geq 1$ for a moment, and set $\overline{\omega} = \omega/(\underline{x}^n)\omega$. Then we have an exact sequence

$$0 \to \operatorname{syz}^d_R(\overline{\omega}) \to F \to \operatorname{syz}^{d-1}_R(\overline{\omega}) \to 0$$

with F a finitely generated free R-module. We apply the functor $-^{\vee} := \operatorname{Hom}_R(-, \omega)$. Note that by [M, 18.2] and [BH, 3.3.5], $\operatorname{Ext}^1_R(\operatorname{syz}^{d-1}_R(\overline{\omega}), \omega) \cong \operatorname{Ext}^d_R(\overline{\omega}, \omega) \cong \operatorname{Hom}_R(\overline{\omega}, \overline{\omega}) \cong R/(\underline{x}^n)$, so we get an exact sequence

$$\operatorname{syz}^d_R(\overline{\omega})^{\vee} \to R/(\underline{x}^n) \to 0.$$

Since R is local, there is an indecomposable direct summand X_n of $\operatorname{syz}^d_R(\overline{\omega})^{\vee}$ mapping onto $R/(\underline{x}^n)$.

The set of isomorphism classes $\{[X_n^{\vee}]\}_{n \geq 1}$ is contained in Σ, so is a finite set. Then $\{[X_n]\}_{n \geq 1}$ is also a finite set. Hence there exists an m such that the indecomposable module X_m maps onto $R/(\underline{x}^n)$ for infinitely many n. Then X_m is free, so $\operatorname{syz}^d_R(\omega/(\underline{x}^m)\omega)^{\vee}$ has a nonzero free direct summand. Dualizing, we see that ω is a direct summand of $\operatorname{syz}^d_R(\omega/(\underline{x}^m)\omega)$. By Lemma 1.3, then, R is Gorenstein on the punctured spectrum. □

We record as a corollary the case of particular interest, from which the main theorem will follow.

Corollary 1.8. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Assume that R has finite syzygy type. Then R is Gorenstein on the punctured spectrum, and in particular has finite CM type.*

Proof. The first statement follows directly from Proposition 1.7. For the second, we apply Lemma 1.3. □

Proof of Main Theorem. Let R be an excellent CM local ring of finite CM type. Then, in particular, R has finite syzygy type. By Proposition 1.5, finite syzygy type ascends to R^{h}.

Greco [G, 5.3] shows that R^{h} is also excellent, and so R^{h} satisfies Artin approximation by [Sw, 2.4]. This implies that R has a canonical module ([Hi], [R]), and so R^{h} is Gorenstein on the punctured spectrum by Corollary 1.8. It follows that R is Gorenstein on the punctured spectrum. Theorem 1.2 finishes the proof. □

We also extend the result of Auslander mentioned in the introduction, that a complete CM local ring of finite CM type is regular on the punctured spectrum, to this more general situation.

Corollary 1.9. *Let $(R, \mathfrak{m})$ be an excellent CM local ring of finite CM type. Then R has an isolated singularity.*

Proof. By the Main Theorem, the completion $\widehat{R}$ also has finite CM type. By Auslander's theorem for complete local rings [A], $\widehat{R}$ has an isolated singularity. This property satisfies faithfully flat descent (see, e.g., [W, 2.7]), so R has an isolated singularity as well. □

2. The non-Cohen-Macaulay case

It follows from the descent criterion of Theorem 1.1 that finite CM type descends to a local ring R from its completion, regardless of whether R is CM. The analogous statement for ascent is false, as the following two examples show. While two may seem like overkill, they fail for different enough reasons that it seems instructive to include both.

Example 2.1. Let $T = k[[x, y, z]]/(x^3 - y^7) \cap (y, z)$, where k is any field. Then T has infinite CM type. To see this, first set $R = k[[x, y]]/(x^3 - x^7)$. Then $R \cong k[[t^3, t^7]]$ has infinite CM type by the classification in [CWW]. Further, $R[[z]]$ has infinite CM type: the map $R \to R[[z]]$ is flat [M, p.53] with CM closed fiber, and Theorem 1.1 applies. Now, $R[[z]] \cong T/(x^3 - y^7)$. It is clear that any two nonisomorphic $R[[z]]$-modules are nonisomorphic as T-modules, so it remains only to see that a MCM $R[[z]]$-module also has depth 2 when viewed as a T-module. This follows from [BH, 1.2.26], so T has infinite CM type.

It is easy to check that the image of x is a nonzerodivisor in T. By [L, Theorem 1], then, T is the completion of some local integral domain A. Then A has finite CM type; in fact, it has no MCM modules at all. For if A had a nonzero MCM module, then A would be universally catenary [Ho, §1]. This implies ([M, p. 252]) that A is formally equidimensional, that is, all minimal primes of T have the same dimension. This is clearly absurd.

Example 2.2. Let k be any field, and let $K = k(t_1, t_2, \dots)$ be an extension of k of infinite transcendence degree. Let f be an irreducible polynomial in n variables over K so that $R_1 = K[x_1, \dots, x_n]_{(x_1,\dots,x_n)}/(f)$ has infinite CM type, and let g be an irreducible polynomial in m variables (with $m \neq n$) so that $R_2 = K[y_1, \dots, y_m]_{(y_1,\dots,y_m)}/(g)$ has infinite CM type. Set

$$R = K[x_1, \dots, x_n, y_1, \dots, y_m]/(f, g).$$

Semilocalize R by inverting the multiplicative set given by the complement of the union of the two ideals $(\underline{x}) = (x_1, \dots, x_n)$ and $(\underline{y}) = (y_1, \dots, y_m)$. Note that these two maximal ideals have different heights. By [dSDL], there exists a subring A of R so that A is a local domain with maximal ideal $(\underline{x}) \cap (\underline{y})$, and $A \hookrightarrow R$ is a finite birational extension. Specifically, note that the two residue fields of R are isomorphic, both being purely transcendental extensions of K. Let ϵ_1 be the surjection $R \to K$ with kernel $(\underline{x})$, and let ϵ_2 be the surjection $R \to K$ with kernel $(\underline{y})$. Then

$$A = \{f \in R \mid \epsilon_1(f) = \epsilon_2(f)\}.$$

The construction in [dSDL] shows that A fails to be catenary: every nonmaximal prime of A has exactly one prime of R lying over it, and the maximal ideal of A is precisely the intersection of the two maximal ideals of R. The preimages of the two saturated chains $0 \subseteq (x_1) \subseteq (x_1, x_2) \subseteq \cdots \subseteq (x_1, \ldots, x_n)$ and $0 \subseteq (y_1) \subseteq (y_1, y_2) \subseteq \cdots \subseteq (y_1, \ldots, y_m)$ give saturated chains of different lengths of primes in A from 0 to the maximal ideal. Thus A fails to be catenary, so has no nonzero MCM modules ([Ho, §1]). In particular, A has finite CM type. We will show that $\widehat{A}$ has infinite CM type. Let $\widehat{R}$ be the completion of R with respect to the Jacobson radical $(\underline{x}) \cap (\underline{y})$. Since the Jacobson radical of R is equal to the maximal ideal of A, we have $\widehat{R} \cong R \otimes_A \widehat{A}$. Since $A \to R$ is birational and finite, there exists a nonzerodivisor $t \in R$ such that $tR \subseteq A$. Then we also have $t\widehat{R} \subseteq \widehat{A}$, so $\widehat{A} \to \widehat{R}$ is also birational and finite. We claim that $\widehat{A}$ has infinite CM type. We can write $\widehat{R} \cong \widehat{R}_1 \times \widehat{R}_2$, so $\widehat{R}$ has infinite CM type (by Theorem 1.1). If two torsion-free $\widehat{R}$-modules are isomorphic as $\widehat{A}$-modules, we can use the birationality to clear denominators and get an $\widehat{R}$-isomorphism. So $\widehat{A}$ has infinite CM type, while A has no MCM modules.

References

[A] M. Auslander, *Isolated singularities and the existence of almost split sequences*, Proc. ICRA IV, Springer Lect. Notes in Math. **1178** (1986), 194-241.

[BGS] R.-O. Buchweitz, G.-M. Greuel and F.-O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities II*, Invent. Math. **88** (1987), 165-182.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Math., vol. 39, Cambridge University Press, Cambridge, 1993.

[CWW] N. Cimen, R. Wiegand and S. Wiegand, *One-dimensional rings of finite representation type*, Abelian Groups and Modules (A. Facchini and C. Menini, eds.), Kluwer, 1995.

[D] S. Ding, *A note on the index of Cohen-Macaulay local rings*, Comm. Alg. **21(1)** (1993), 53-71.

[DI] F. DeMeyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Lecture Notes in Mathematics, vol. 181, Springer-Verlag, Berlin, 1971.

[dSDL] A. M. de Souza Doering and Ives Lequain, *The gluing of maximal ideals–applications to a problem in the spectrum of a Noetherian ring and to problems on the going up and goind down in polynomial rings /jour Trans. Amer. Math. Soc.* **260** (1980), 583-593.

[EG] E. G. Evans and P. Griffith, *Syzygies*, London Math. Soc. Lect. Notes **106**, 1985.

[G] S. Greco, *Two theorems on excellent rings*, Nagoya Math. J. **60** (1970), 139-149.

[GK] G.-M. Greuel and H. Knörrer, *Einfache Kurvensingularitäten und torsionfreie Moduln*, Math. Ann. **270** (1985), 417-425.

[GKr] G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), 229–354.

[Gu] R. Guralnick, *Lifting homomorphisms of modules*, Ill. J. Math **29 (1)** (1985), 153-156.

[Hi] V. Hinich, *Rings with the approximation property admit a dualizing complex*, Math. Nachr. **163** (1993), 289-296.

[Ho] M. Hochster, *Cohen-Macaulay modules*, Conference on Commutative Algebra: Lawrence, Kansas 1972 (J. W. Brewer and Edgar A. Rutter, eds.), Lecture Notes in Mathematics, vol. 311, Springer-Verlag, Berlin, 1973, pp. 120-152.

[I] B. Iverson, *Generic Local Structure of the Morphisms in Commutative Algebra*, Lecture Notes in Mathematics, vol. 310, Springer-Verlag, Berlin, 1973.

[K] H. Knörrer, *Cohen-Macaulay modules on hypersurface singularities I /jour Invent. Math.* **88** (1987), 153-164.

[L] C. Lech, *A method for constructing bad Noetherian local rings*, Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Mathematics, vol. 1183, Springer-Verlag, Berlin, 1986, pp. 241-247.

[M] H. Matsumura, *Commutative Algebra*, W. A. Benjamin, New York, 1970.

[R] C. Rotthaus, *Descent of the canonical module in rings with the approximation property*, Proc. Amer. Math. Soc. **124 (6)** (1996), 1713-1717.

[RWW] C. Rotthaus, D. Weston, and R. Wiegand, *Indecomposable Gorenstein modules of odd rank*, J. Algebra **214** (1999), 122–127.

[Sc] F.-O. Schreyer, *Finite and countable CM-representation type*, Singularities, Representation of Algebras, and Vector Bundles: Proceedings Lambrecht 1985 (G.-M. Greuel and G. Trautmann, eds.), Lecture Notes in Mathematics, vol. 1273, Springer-Verlag, Berlin, 1987, pp. 9-34.

[So] Ø. Solberg, *Hypersurface singularities of finite Cohen-Macaulay type*, Proc. London Math. Soc. **58** (1989), 258–280.

[Sw] R. Swan, *Néron-Popescu desingularization*, preprint.

[W] R. Wiegand, *Local rings of finite Cohen-Macaulay type*, J. Algebra **203** (1998), 156-168.

[Y] Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Math. Soc. Lect. Notes **146**, 1990.