Lecture 3. Resolutions and derived functors (GL)

This lecture is intended to be a whirlwind introduction to, or review of, resolutions and derived functors – with tunnel vision. That is, we'll give unabashed preference to topics relevant to local cohomology, and do our best to draw a straight line between the topics we cover and our final goals. At a few points along the way, we'll be able to point generally in the direction of other topics of interest, but other than that we will do our best to be single-minded.

Appendix A contains some preparatory material on injective modules and Matlis theory. In this lecture, we will cover roughly the same ground on the projective/flat side of the fence, followed by basics on projective and injective resolutions, and definitions and basic properties of derived functors.

Throughout this lecture, let us work over an unspecified commutative ring R with identity. Nearly everything said will apply equally well to noncommutative rings (and some statements need even less!).

In terms of module theory, fields are the simple objects in commutative algebra, for all their modules are *free*. The point of resolving a module is to measure its complexity against this standard.

Definition 3.1. A module F over a ring R is *free* if it has a *basis*, that is, a subset $B \subseteq F$ such that B generates F as an R-module and is linearly independent over R.

It is easy to prove that a module is free if and only if it is isomorphic to a direct sum of copies of the ring. The cardinality of a basis S is the *rank* of the free module. (To see that the rank is well-defined, we can reduce modulo a maximal ideal of R and use the corresponding result for—what else?—fields.)

In practice and computation, we are usually satisfied with free modules. Theoretically, however, the properties that concern us are *projectivity* and *flatness*. Though the definition of freeness given above is "elementary", we could also have given an equivalent definition in terms of a universal lifting property. (It's a worthwhile exercise to formulate this property, and you'll know when you've got the right one because the proof is trivial.) For projective modules, we reverse the process and work from the categorical definition to the elementary one.

Definition 3.2. An R-module P is *projective* if whenever there exist a surjective homomorphism of R-modules $f : M \longrightarrow N$ and an arbitrary homomorphism of R-modules $g : P \longrightarrow N$, there is a lifting $h : P \longrightarrow M$ so that $fh = g$. Pictorially, we have

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle g} & & \\ M & \underset{f}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

with the bottom row forming an exact sequence of R-modules.

Here is another way to word the definition which highlights our intended uses for projective modules. Let $\mathcal{F}$ be a *covariant* functor from R-modules to abelian groups. Recall that $\mathcal{F}$ is said to be *left-exact* if for each short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\,,$$

there is a corresponding induced exact sequence

$$0 \longrightarrow \mathcal{F}(M') \longrightarrow \mathcal{F}(M) \longrightarrow \mathcal{F}(M'') .$$

If in addition the induced map $\mathcal{F}(M) \longrightarrow \mathcal{F}(M'')$ is surjective, then we say that $\mathcal{F}$ is *exact*. It is easy to show that for any R-module N, $\operatorname{Hom}_R(N, -)$ is a covariant left-exact functor.

Exercise 3.3. Prove that P is projective if and only if $\operatorname{Hom}_R(P, -)$ is exact.

Here are the first four things that you should check about projectives, plus one.

(1) Free modules are projective.
(2) A module P is projective if and only if there is a module Q such that $P \oplus Q$ is free.
(3) Arbitrary direct sums of projective modules are projective.
(4) Freeness and projectivity both localize.
(5) Over a Noetherian local ring R, all projectives are free.

Example 3.4. Despite their relatively innocuous definition, projective modules are even now a very active area of research. Here are a couple of highlights.

(1) Let R be a polynomial ring over a field $\mathbb{K}$. Then all finitely generated projective R-modules are free. This is the content of the rightly renowned Quillen-Suslin theorem [119, 142], also known as Serre's Conjecture, pre-1978 (see [5]). It's less well-known that the Quillen-Suslin theorem holds as well when $\mathbb{K}$ is a discrete valuation ring. Closely related is the Bass-Quillen Conjecture, which asserts for any regular ring R that every projective module over $R[T]$ is extended from R. Quillen and Suslin's solutions of Serre's Conjecture proceed by proving this statement when R is a regular ring of dimension at most 1. Popescu's celebrated theorem of "General Néron Desingularization" [118, 143], together with results of Lindel [92], proves Bass-Quillen for regular local rings $(R, \mathfrak{m})$ such that either R contains a field, $\operatorname{char}(R/\mathfrak{m}) \notin \mathfrak{m}^2$, or R is excellent and Henselian.
(2) In the ring $R = \mathbb{Z}[\sqrt{-5}]$, the ideal $\mathfrak{a} = (3, 2 + \sqrt{-5})$ is projective but not free as an R-module. Indeed, $\mathfrak{a}$ is not principal, so cannot be free (prove this!), while the obvious surjection $R^2 \longrightarrow \mathfrak{a}$ has a splitting given by
$$x \mapsto x \cdot \left(\frac{-1 + \sqrt{-5}}{2 + \sqrt{-5}}, \frac{2 - \sqrt{-5}}{3} \right)$$
so that $\mathfrak{a}$ is a direct summand of R^2. (This is of course directly related to the fact that R is not a UFD.)
(3) Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, the coordinate ring of the real 2-sphere. Then the homomorphism $R^3 \longrightarrow R$ defined by the row vector $\nu = [x, y, z]$ is surjective, so the kernel P satisfies $P \oplus R \cong R^3$. However, it can be shown that P is not free. Every element of R^3 gives a vector field in $\mathbb{R}^3$, with ν defining the vector field pointing straight out from the origin. An element of P thus gives a vector field that is tangent to the 2-sphere in $\mathbb{R}^3$. If P were free, a basis would define two linearly independent vector fields on the 2-sphere. But hedgehogs can't be combed!

As we noted above, the definition of projectivity amounts to saying that some usually half-exact functor is exact. You can also check easily that an R-module I

is *injective* if and only if the *contravariant* functor $\mathrm{Hom}_R(-, I)$ is exact. Our next step is to mimic these two statements for the other half-exact functor that we're familiar with.

Recall that for a given R-module M, the functor $- \otimes_R M$ is right-exact. (The proof is "elementary", in that the best way to approach it is by chasing elements.) It's clear that if we take $M = R$, then $A \otimes_R M$ and $B \otimes_R M$ are nothing but A and B again, so that in fact $- \otimes_R R$ is exact. With an eye toward defining the Tor and Ext functors below, we give this property its rightful name.

Definition 3.5. An R-module M is *flat* provided $- \otimes_R M$ is an exact functor.

We have already observed that the free module R is flat, and it is easy to check that the direct sum of a family of flat modules is flat. Thus free modules are trivially flat, and it follows immediately from the distributivity of $\otimes$ over $\oplus$ that projective modules are flat as well. In fact, it is very nearly true that the only flat modules are the projectives. Specifically,

Theorem 3.6 (Govorov and Lazard [43, 87], see [28])**.** *An R-module M is flat if and only if M is a direct limit of a directed system*[6] *of free modules. In particular, a finitely generated flat module is projective.*

Having defined the three classes of modules to which we will compare all others, let us move on to resolutions.

Definition 3.7. Let M be an R-module.

- An *injective resolution* of M is an exact sequence of the form
$$E^\bullet : 0 \longrightarrow M \longrightarrow E^0 \xrightarrow{\varphi^1} E^1 \xrightarrow{\varphi^2} E^2 \longrightarrow \cdots$$
with each E^n injective.
- A *projective resolution* of M is an exact sequence of the form
$$P_\bullet : \cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow M \longrightarrow 0$$
with each P_n projective.
- A *flat resolution* of M is an exact sequence of the form
$$F_\bullet : \cdots \longrightarrow F_2 \xrightarrow{\rho_2} F_1 \xrightarrow{\rho_1} F_0 \longrightarrow M \longrightarrow 0$$
with each F_n flat.

Remark 3.8. Each of the resolutions above exist for any R-module M; another way to say this is that the category of R-modules *has enough projectives* and *enough injectives*. (Since projectives are flat, there are of course also enough flats.) In contrast, the category of sheaves over projective space does *not* have enough projectives, as we'll see in Lecture 12!

Slightly more subtle is the question of minimality. Let us deal with injective resolutions first. We say that $E^\bullet$ as above is a *minimal* injective resolution if each E^n is the injective hull of the image of $\varphi^n : E^{n-1} \longrightarrow E^n$. As in the proof of Theorem A.21, we see that E is an injective hull for a submodule M if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$, the map $\mathrm{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \longrightarrow \mathrm{Hom}_R(R/\mathfrak{p}, E)_\mathfrak{p}$ is an isomorphism. Therefore, $E^\bullet$ is a *minimal* injective resolution if and only if the result of applying $\mathrm{Hom}_R(R/\mathfrak{p}, -)_\mathfrak{p}$ to each homomorphism in $E^\bullet$ is the zero map.

[6] For "direct limit of a directed system" in this statement, you can substitute "union of submodules" without too much loss of sense. For more on direct limits, see Lecture 3.

The *injective dimension* of M, $\mathrm{id}_R M$, is the minimal length of an injective resolution of M. (If no resolution of finite length exists, we say $\mathrm{id}_R M = \infty$.) We have $\mathrm{id}_R M = 0$ if and only if M is injective. Theorem A.25 shows that not only is this concept well-defined, it can be determined in terms of the Bass numbers of M. Observe that all this bounty springs directly from the structure theory of injective modules over Noetherian rings, Theorem A.21.

In contrast, the theory of minimal projective resolutions works best over local rings R, where, not coincidentally, all projective modules are free. See Lecture 7 for more in this direction. In any case, we define the *projective dimension* of M, $\mathrm{pd}_R\, M$, as the minimal length of a projective resolution of M, or ∞ if no finite resolution exists.

Finally, for completeness, we mention that *flat (or weak) dimension* is the minimal length of a flat resolution. For finitely generated modules over Noetherian rings, this turns out to be exactly the same as projective dimension, so we won't have much need for it.

One main tool for proving existence and uniqueness of derived functors will be the following Comparison Theorem. It comes in two dual flavors, the proof of each being immediate from the definitions.

Theorem 3.9 (Comparison Theorem)**.** *Let $f : M \longrightarrow N$ be a homomorphism of R-modules.*

(1) Assume that we have a diagram

$$\begin{array}{llllllllllll} J^\bullet : & 0 & \longrightarrow & M & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\ & & & \downarrow f & & & & & & & & \\ I^\bullet : & 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \end{array}$$

with $J^\bullet$ exact and $I^\bullet$ a complex of injective modules. Then there is a lifting $\varphi^\bullet : J^\bullet \longrightarrow I^\bullet$ of f, and $\varphi^\bullet$ is unique up to homotopy.

(2) Assume that we have a diagram of homomorphisms of R-modules

$$\begin{array}{llllllllllll} P_\bullet : & \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & & & & \downarrow f & & \\ Q_\bullet : & \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with $P_\bullet$ a complex of projective modules, and $Q_\bullet$ exact. Then there is a lifting $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ of f, and $\varphi_\bullet$ is unique up to homotopy.

Recall that two degree-zero maps of complexes $\varphi_\bullet, \psi_\bullet : (F_\bullet, \partial^F) \longrightarrow (G_\bullet, \partial^G)$ are *homotopic* (or *homotopy-equivalent*) if there is a map of degree -1, $s : F_\bullet \longrightarrow G_\bullet$, so that

$$\varphi_\bullet - \psi_\bullet = \partial^G s - s\partial^F .$$

Exercise 3.10. Prove that homotopic maps induce the same homomorphism in homology.

At last we define derived functors. The basic strategy is as follows: for a half-exact additive functor $\mathcal{F}$ and module M, resolve M by modules that are *acyclic* for $\mathcal{F}$, apply $\mathcal{F}$ to the complex obtained by deleting M from the resolution, and take (co)homology. The details vary according to whether $\mathcal{F}$ is left- or right-exact and co- or contravariant. We give here the one most relevant to our purposes, and leave it to the reader to formulate the others.

Definition 3.11. Let $\mathcal{F}$ be an additive, covariant, left-exact functor (for example, $\mathrm{Hom}_R(M, -)$ for some fixed R-module M). Let $M \longrightarrow E^\bullet$ be an injective resolution. Then $\mathcal{F}(E^\bullet)$ is a complex; the i^{th} right derived functor of $\mathcal{F}$ on M is defined by $R^i\mathcal{F}(M) := H^i(\mathcal{F}(E^\bullet))$.

Remark 3.12. Derived functors, both the flavor defined above and the corresponding ones for other variances and exactnesses, satisfy appropriate versions of the following easily-checked properties. Let $\mathcal{F}$ be as in Definition 3.11. Then

(1) $R^i\mathcal{F}$ is well-defined up to isomorphism (use the Comparison Theorem). More generally, any homomorphism $f : M \longrightarrow N$ gives rise to homomorphisms $R^i\mathcal{F}(f) : R^i\mathcal{F}(M) \longrightarrow R^i\mathcal{F}(N)$ for every $i \geq 0$. In particular, if $\mathcal{F}$ is multiplicative (so that $\mathcal{F}$ takes multiplication by $r \in R$ to multiplication by r), then so is $R^i\mathcal{F}$.
(2) $R^0\mathcal{F} = \mathcal{F}$, and $R^i\mathcal{F}(E) = 0$ for all $i > 0$ if E is injective.
(3) For every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, there are *connecting homomorphisms* δ^i and a long exact sequence

$$\cdots \longrightarrow R^i\mathcal{F}(M) \longrightarrow R^i\mathcal{F}(M'') \xrightarrow{\delta^i} R^{i+1}\mathcal{F}(M') \longrightarrow R^{i+1}\mathcal{F}(M) \longrightarrow \cdots.$$

For our purposes, there are three main examples of derived functors. We define two of them here; the third will make its grand entrance in Lecture 6.

Definition 3.13. Let M and N be R-modules.

(1) The Ext *functors* $\mathrm{Ext}^i_R(M, N)$, $i \geq 0$, are the right derived functors of $\mathrm{Hom}_R(M, -)$.
(2) The Tor *functors* $\mathrm{Tor}^R_i(M, N)$, $i \geq 0$, are the left derived functors of $- \otimes_R N$.

A sharp eye might see that we've smuggled a few theorems in with this definition. There are two potential descriptions of Ext: while we chose to use the right derived functors of the left-exact covariant functor $\mathrm{Hom}_R(M, -)$, we could also have used the right derived functors of the left-exact *contravariant* functor $\mathrm{Hom}_R(-, N)$. More concretely, our definition gives the following recipe for computing Ext: Given M and N, let $I^\bullet$ be an injective resolution of N, and compute $\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}_R(M, I^\bullet))$. An alternative definition would proceed by letting $P_\bullet$ be a projective resolution of M, and computing $\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}_R(P_\bullet, N))$. It is a theorem (which we will not prove) that the two approaches agree. Similarly, $\mathrm{Tor}^R_i(M, N)$ can be computed either by applying $M \otimes_R -$ to a flat resolution of N, or by applying $- \otimes_R N$ to a flat resolution of M.

Here are two examples of computing Tor and Ext.[7]

[7] It's possible that these examples are too namby-pamby. Another possibility would be to replace them by the 0134 and 2-by-3 examples. I'm open to suggestions.

Example 3.14. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x, y, z]$. Denote the residue field $R/(x, y, z)R$ again by $\mathbb{K}$. We assert that

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}} R^3 \xrightarrow{[\,x\ y\ z\,]} R \longrightarrow 0$$

is a (truncated) free resolution of $\mathbb{K}$. (You can check this directly and laboriously, or wait until Lectures 5 and 7.) From it we can calculate $\operatorname{Tor}_i^R(\mathbb{K}, \mathbb{K})$ and $\operatorname{Ext}_R^i(\mathbb{K}, R)$ for all $i \geq 0$. For the Tor_i, we apply $- \otimes_R \mathbb{K}$ to the resolution. Each free module R^b becomes $R^b \otimes_R \mathbb{K} \cong \mathbb{K}^b$, and each matrix is reduced modulo the ideal (x, y, z). The result is the complex

$$0 \longrightarrow \mathbb{K} \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K} \longrightarrow 0$$

with zero differentials at every step. Thus

$$\operatorname{Tor}_i^R(\mathbb{K}, \mathbb{K}) \cong \begin{cases} \mathbb{K} & \text{for } i = 0; \\ \mathbb{K}^3 & \text{for } i = 1; \\ \mathbb{K}^3 & \text{for } i = 2; \\ \mathbb{K} & \text{for } i = 3; \\ 0 & \text{for } i \geq 4. \end{cases}$$

Applying $\operatorname{Hom}_R(-, R)$ has the effect of replacing each matrix in our resolution of $\mathbb{K}$ by its transpose, which yields

$$0 \longleftarrow R \xleftarrow{[\,x\ y\ z\,]} R^3 \xleftarrow{\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}} R^3 \xleftarrow{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} R \longleftarrow 0\,.$$

Noting the striking similarity of this complex to the one we started with, we conclude that

$$\operatorname{Ext}_R^i(\mathbb{K}, R) \cong \begin{cases} \mathbb{K} & \text{if } i = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.15. Let $\mathbb{K}$ again be a field and put $R = \mathbb{K}[x, y]/(xy)$. Set $M = R/(x)$ and $N = R/(y)$. To compute Tor_i and Ext^i, let us start with a projective resolution of M. As the kernel of multiplication by x is the ideal y, and vice versa, we obtain the free resolution

$$F_\bullet : \ \cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow M \longrightarrow 0\,.$$

Computing $\operatorname{Tor}_i^R(M, N)$ requires that we truncate $F_\bullet$ and apply $- \otimes_R N$. In effect, this replaces each copy of R by $N = R/(y) \cong \mathbb{K}[x]$:

$$F_\bullet : \ \cdots \xrightarrow{x} R/(y) \xrightarrow{y} R/(y) \xrightarrow{x} R/(y)\,.$$

Since y kills $R/(y)$ while x is a nonzerodivisor on $R/(y)$, computing kernels and images quickly reveals that

$$\operatorname{Tor}_i^R(M, N) \cong \begin{cases} \mathbb{K} & \text{for } i \geq 0 \text{ even, and} \\ 0 & \text{for } i \geq 0 \text{ odd.} \end{cases}$$

Similarly, applying $\mathrm{Hom}_R(-,N)$ replaces each R by $N = R/(y)$, but this time reverses all the arrows:

$$\mathrm{Hom}_R(F_\bullet, N): \; R/(y) \xrightarrow{x} R/(y) \xrightarrow{y} R/(y) \xrightarrow{x} \cdots .$$

We see that

$$\mathrm{Ext}^i_R(M,N) \cong \begin{cases} 0 & \text{for } i \geq 0 \text{ even, and} \\ \mathbb{K} & \text{for } i \geq 0 \text{ odd.} \end{cases}$$

Finally, apply $\mathrm{Hom}_R(-,R)$ to find that

$$\mathrm{Ext}^i_R(M,R) \cong \begin{cases} N & \text{for } i = 0 \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

We finish this section with the main properties of Ext and Tor that we'll use repeatedly in the lectures to follow. They follow directly from the properties of derived functors listed above.

Theorem 3.16. *Let R be a ring and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a short exact sequence of R-modules. Then for any R-module N, there are three long exact sequences

$$\cdots \to \mathrm{Ext}^i_R(N,M) \to \mathrm{Ext}^i_R(N,M'') \to \mathrm{Ext}^{i+1}_R(N,M') \to \mathrm{Ext}^{i+1}_R(N,M) \to \cdots$$

$$\cdots \to \mathrm{Ext}^{i+1}_R(M,N) \to \mathrm{Ext}^{i+1}_R(M',N) \to \mathrm{Ext}^i_R(M'',N) \to \mathrm{Ext}^i_R(N,M) \to \cdots$$

$$\cdots \to \mathrm{Tor}^{i+1}_R(M,N) \to \mathrm{Tor}^{i+1}_R(M'',N) \to \mathrm{Tor}^i_R(M',N) \to \mathrm{Tor}^i_R(M,N) \to \cdots$$

Theorem 3.17. *Let M be an R-module. Each of the following three sets of conditions are equivalent:*

- *(1a)* *M is injective;*
- *(1b)* $\mathrm{Ext}^i_R(-,M) = 0$ *for all $i \geq 1$;*
- *(1c)* $\mathrm{Ext}^1_R(-,M) = 0$.
- *(2a)* *M is projective;*
- *(2b)* $\mathrm{Ext}^i_R(M,-) = 0$ *for all $i \geq 1$;*
- *(2c)* $\mathrm{Ext}^1_R(M,-) = 0$.
- *(3a)* *M is flat;*
- *(3b)* $\mathrm{Tor}^i_R(-,M) = 0$ *for all $i \geq 1$;*
- *(3c)* $\mathrm{Tor}^1_R(-,M) = 0$.

Lecture 5. Complexes from a sequence of ring elements (GL)

In Lecture 3 we postulated or proved the existence of several examples of exact sequences: projective, free, or injective resolutions, as well as long exact sequences in (co)homology. Most of these were quite abstract, and the concrete examples came out of thin air. The problem of actually producing any one of these kinds of resolutions for a given module was essentially ignored. In this lecture, we will give a few, quite concrete, constructions of *complexes* beginning from an explicit list of ring elements, which we can later use and manipulate to obtain resolutions in some cases. The complexes we construct will also, as we shall see, carry quite a lot of information that is relevant to our long-term goal of understanding local cohomology.

In constructing various complexes from a sequence of elements, we will begin with the case of a single element, and inductively patch copies together to build the final product. This patching will be done by taking the tensor product of two or more complexes, a procedure we now define in general. Let R be an arbitrary commutative ring.

Definition 5.1. Let

$$F^\bullet : \cdots \longrightarrow F^i \xrightarrow{\varphi^i} F^{i+1} \longrightarrow \cdots$$

and

$$G^\bullet : \cdots \longrightarrow G^i \xrightarrow{\psi^i} G^{i+1} \longrightarrow \cdots$$

be (cohomologically indexed) complexes of R-modules. Then the *tensor product* of F and G is

$$F^\bullet \otimes_R G^\bullet : \cdots \longrightarrow \bigoplus_{i+j=k} F^i \otimes_R G^j \xrightarrow{\partial^k} \bigoplus_{i+j=k+1} F^i \otimes_R G^j \longrightarrow \cdots,$$

where ∂^k is defined on simple tensors $x \otimes y \in F^i \otimes_R G^j$ by

$$\partial^k(x \otimes y) = \varphi^i(x) \otimes y + (-1)^i x \otimes \psi^j(y).$$

An exactly similar definition applies to homologically-indexed complexes.

Remark 5.2. The sign in the definition of ∂^k is there precisely so that $F^\bullet \otimes_R G^\bullet$ is a complex. With this definition, it is straightforward to check that the tensor product defines an honest binary operation on complexes, which, if R is commutative, is both associative and commutative.

The Koszul complex

When we are handed a single element of a ring, there is one complex simply crying out to be constructed.

Definition 5.3. Let R be a ring and $x \in R$. The *Koszul complex on x* is

$$K_\bullet(x) : \quad 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0,$$

with R in degrees 1 and 0. For a sequence $\underline{x} = x_1, \dots, x_n$, the Koszul complex on $\underline{x}$ is defined by

$$K_\bullet(\underline{x}) = K_\bullet(x_1) \otimes_R \cdots \otimes_R K_\bullet(x_n).$$

Example 5.4. Let $x, y \in R$. The Koszul complex on x is

$$K_\bullet(x): \quad 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0\,,$$

and that on y is

$$K_\bullet(y): \quad 0 \longrightarrow R \xrightarrow{\ y\ } R \longrightarrow 0\,.$$

The tensor product is

$$K_\bullet(x,y): \quad 0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]} R^2 \xrightarrow{[y\ \ x]} R \longrightarrow 0$$

where the three nonzero modules are in degrees 2, 1, and 0, left to right. Observe that $K_\bullet(x,y)$ is indeed a complex.

Remark 5.5. Let $\underline{x} = x_1, \ldots, x_n$. Then some simple counting using Definition 5.1 reveals that the r^{th} module in the Koszul complex $K_\bullet(\underline{x})$ is given by

$$K_r(\underline{x}) \cong R^{\binom{n}{r}}\,,$$

where $\binom{n}{r}$ is the appropriate binomial coefficient. The natural basis for this free module is the set $\{e_{i_1,\ldots,i_r}\}$, where $1 \le i_1 < \cdots < i_r \le n$. In terms of this basis, the r^{th} differential ∂_r is given by

$$\partial_r(e_{i_1,\ldots,i_r}) = \sum_{j=1}^{r} (-1)^{i-1} x_{i_j} e_{i_1,\ldots,\widehat{i_j},\ldots,i_r}\,.$$

Exercise 5.6. Construct the Koszul complex on a sequence of three elements $x, y, z \in R$. Compare with Example 3.14.

Exercise 5.7. For a sequence of any length, $\underline{x} = x_1, \ldots, x_n$, identify the maps ∂_1 and ∂_n in $K_\bullet(\underline{x})$.

The Koszul complex as defined above holds an enormous amount of information about the sequence $\underline{x} = x_1, \ldots, x_n$ and the ideal of R that they generate. In future lectures we'll see some of this information laid bare. For the best applications, however, we will want more *relative* information about $\underline{x}$ and its impact on various R-modules. We therefore define the Koszul complex on a module M, and introduce the Koszul homology groups.

Definition 5.8. Let R be a commutative ring, $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R, and M an R-module.

(1) The *Koszul complex of* $\underline{x}$ *on* M is $K_\bullet(\underline{x}, M) := K_\bullet(\underline{x}) \otimes_R M$.
(2) The *Koszul homology of* $\underline{x}$ *on* M is the homology of this complex, so $H_j(\underline{x}, M) := H_j(K_\bullet(\underline{x}, M))$ for $j = 0, \ldots, n$.

Example 5.9. Let $x \in R$ be a single element and M an R-module. Then the tininess of the Koszul complex $K_\bullet(x, M)$ makes computing the Koszul homology trivial:

$$H_0(x,M) = M/xM$$
$$H_1(x,M) = (0 :_M x) = \{m \in M \mid xm = 0\}.$$

In particular, we can make two immediate observations:

(1) If $xM \neq M$, that is, x does not act "like a unit" on M, then $H_0(x, M) \neq 0$. In particular, if M is finitely generated and x is in the Jacobson radical of R, then H_0 is nonzero by Nakayama's Lemma.

(2) If x is a *nonzerodivisor* on M, that is, $xm \neq 0$ for all nonzero $m \in M$, then $H_1(x, M) = 0$.

In order to put this example in its proper context, let us insert here a brief interlude on *regular sequences* and *depth*.

Regular sequences and depth: a first look

Definition 5.10. Let R be a ring and $x \in R$. We say that x is a *nonzerodivisor* if $xy \neq 0$ for all nonzero $y \in R$. If in addition x is a nonunit, say that x is a *regular element*.

Let moreover M be an R-module. Then x is a *nonzerodivisor on* M if $xm \neq 0$ for all nonzero $m \in M$, and a *regular element on* M (or M*-regular*) if in addition $xM \neq M$.

Remark 5.11. From Example 5.9 we see that $x \in R$ is M-regular if and only if $H_0(x, M) \neq 0$ and $H_1(x, M) = 0$.

Definition 5.12. A sequence of elements $\underline{x}$ of R is a *regular sequence on* M if

(1) x_1 is M-regular, and
(2) for each $i = 2, \ldots, n$, x_i is regular on $M/(x_1, \ldots, x_{i-1})M$.

Remark 5.13.

(1) Some authors allow the possibility that $xM = M$, and call such a sequence "weakly" M-regular.
(2) In such an inductive definition, the order of the x_i is essential. For example, the sequence $X - 1, XY, XZ$ is regular in $\mathbb{K}[X, Y, Z]$, while $XY, XZ, X - 1$ is not. If, however, $(R, \mathfrak{m})$ is local and $\underline{x}$ is contained in $\mathfrak{m}$, then we shall see below that the order of the x_i is immaterial.

The set of regular elements is easy to describe.

Lemma 5.14. *Let R be a Noetherian commutative ring. The set of zerodivisors on a finitely generated R-module M is the union of the associated primes of M.*

Proof. Exercise. □

We'll finish this interlude by smuggling in one more definition.

Definition 5.15. Let $\mathfrak{a} \subseteq R$ be an ideal and M an R-module. The *depth of* $\mathfrak{a}$ *on* M is the maximal length of an M-regular sequence contained in $\mathfrak{a}$, denoted $\operatorname{depth}_R(\mathfrak{a}, M)$.

Depth will reappear in Lectures 7 and 8.

Back to the Koszul complex

Let us return now to the Koszul complex. We computed above the Koszul homology of a single element, and now recognize it as determining regularity. We have high hopes for the case of two elements.

Example 5.16. Let $x, y \in R$ and let M be an R-module. Then the homology of the complex

$$K_\bullet(\{x, y\}, M): \ 0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]} M^2 \xrightarrow{[\,y\ x\,]} M \longrightarrow 0$$

at the ends can be computed easily. We have

$$H_0(\{x,y\},M) = M/(x,y)M, \text{ and}$$
$$H_2(\{x,y\},M) = (0 :_M (x,y)) = \{m \in M \mid xm = ym = 0\}.$$

What is the homology in the middle? Let $(a,b) \in M^2$ be such that $ya + bx = 0$. Then $ya = -xb$, so in particular $a \in (xM :_M y)$. Assume for the moment that x is regular on M, and take $a \in (xM :_M y)$. Then there exists some b so that $ya = -xb$, and since x is M-regular, there is *precisely one* such b. In other words, if we assume that x is a nonzerodivisor on M, then $\ker[x\ y] \cong (xM :_M y)$. Still assuming that x is M-regular, we can also identify the image of $\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$ as

$$\{(xc, -yc) \mid c \in M\} \cong xM\,.$$

Therefore

$$H_1(\{x,y\},M) \cong (xM :_M y)/xM\,.$$

Exercise 5.17. Assume that x is M-regular, and prove that x, y is an M-regular sequence if and only if $(xM :_M y) = xM$.

Examples 5.16 and 5.11 are part of what is usually called the "depth-sensitivity" of the Koszul complex. See [104, 16.5] for a proof.

Theorem 5.18. *Let R be a commutative ring and $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R. If $\underline{x}$ is regular on M, then $H_j(\underline{x}, M) = 0$ for all $j > 0$ and $H_0(\underline{x}, M) = M/\underline{x}M \neq 0$. If either $(R, \mathfrak{m})$ is Noetherian local, $x \in \mathfrak{m}$, and M is nonzero finitely generated, or R is $\mathbb{N}$-graded, M is nonzero $\mathbb{N}$-graded, and the elements $\underline{x}$ are homogeneous of positive degree, then there is a strong converse: if $H_1(\underline{x}, M) = 0$, then $\underline{x}$ is an M-regular sequence.*

As a corollary, we can conclude that depth is a "geometric" property:

Corollary 5.19. *If $x_1, \ldots, x_n$ is an M-regular sequence, then $x_1^{a_1}, \ldots x_n^{a_n}$ is M-regular as well for any positive integers $a_1, \ldots, a_n$. In particular,* $\operatorname{depth}_R(\mathfrak{a}, M) = \operatorname{depth}_R(\sqrt{\mathfrak{a}}, M)$.

We also mention the following fact, which we won't need, but which motivates some of our results in Lecture 7. See [28, 17.4] for a proof.

Proposition 5.20. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. Suppose that $\mathfrak{a}$ is minimally generated by n elements and that $\mathfrak{a}$ contains an M-regular sequence of length n. Then any minimal system of generators for $\mathfrak{a}$ is an M-regular sequence.*

For later applications, it will occasionally be useful to adjust the indexing of the Koszul complex.

Definition 5.21. Let R be a commutative ring and $x \in R$. The *cohomological Koszul complex on x* is

$$K^\bullet(x): \quad 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0\,,$$

which is identical to the usual Koszul complex, except with R in degrees 0 and 1. For a sequence $\underline{x} = x_1, \ldots, x_n$, define $K^\bullet(\underline{x}) = K^\bullet(x_1) \otimes_R \cdots \otimes_R K^\bullet(x_n)$. Finally, for an R-module M, we put $K^\bullet(\underline{x}, M) = K^\bullet(\underline{x}) \otimes_R M$.

Exercise 5.22. Prove that $K^\bullet(\underline{x}) \otimes_R M$ is isomorphic to $\operatorname{Hom}_R(K_\bullet(\underline{x}), M)$.

The Čech complex

Given again a single element x in a ring R, it may seem like the only complex we can build from such meager information is the Koszul complex. If we insist on clinging to the world of finitely generated R-modules, this is essentially true. If, however, we allow some small amount of infinite generation, new vistas open to us.

The Čech complex attached to a sequence of ring elements, like the Koszul, is built inductively by tensoring together short complexes. Recall that for $x \in R$, the localization R_x, also sometimes written $R[\frac{1}{x}]$, is obtained by inverting the multiplicatively closed set $\{1, x, x^2, \dots\}$.

Definition 5.23. Let R be a commutative ring and $x \in R$. The *Čech complex on* x is

$$C^\bullet(x;R): \quad 0 \longrightarrow R \xrightarrow{\iota} R_x \longrightarrow 0\,,$$

with R in degree 0 and R_x in degree 1, and where ι is the canonical map sending each $r \in R$ to the fraction $\frac{r}{1} \in R_x$. For a sequence $\underline{x} = x_1, \dots, x_n$ in R, the Čech complex on $\underline{x}$ is $C^\bullet(\underline{x};R) := C^\bullet(x_1;R) \otimes_R \cdots \otimes_R C^\bullet(x_n;R)$. For an R-module M, define $C^\bullet(\underline{x};M) = C^\bullet(\underline{x};R) \otimes_R M$. The j^{th} *Čech cohomology* is defined by $H^j(\underline{x};M) := H^j(C^\bullet(\underline{x};M))$.

Example 5.24. As with the Koszul complexes, the Čech complex is easy to describe for small n. In case $\underline{x} = x$ is a single element, $C^\bullet(x;R)$ is given by the definition. We note that

$$\begin{aligned} H^0(x;R) &= \{r \in R \mid \frac{r}{1} = 0 \text{ in } R_x\} \\ &= \{r \in R \mid x^a r = 0 \text{ for some } a \geq 0\} \\ &= \bigcup_{a \geq 0} 0 :_R x^a \end{aligned}$$

is the union of annihilators of x^a. This is sometimes written $0 :_R x^\infty$.

Meanwhile, $H^1(x;R) \cong R_x/R$. This expression for H^1 is ambiguous and not very satisfying (particularly if x is a zerodivisor); we'll correct for this shortly. For now, suppose that $R = \mathbb{K}[x]$ is the univariate polynomial ring over a field $\mathbb{K}$. Then $R_x \cong \mathbb{K}[x,x^{-1}]$ is the ring of *Laurent polynomials*. The quotient $\mathbb{K}[x,x^{-1}]/\mathbb{K}[x]$ is generated (over $\mathbb{K}$) by all the negative powers x^{-c}, $c \in \mathbb{N}$, and has R-module structure dictated by

$$x^a x^{-c} = \begin{cases} x^{a-c} & \text{if } a < c, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 5.25. Prove that $\mathbb{K}[x,x^{-1}]/\mathbb{K}[x]$ is isomorphic to the injective hull of the residue field of $\mathbb{K}[x]$.

Example 5.26. For $\underline{x} = \{x,y\}$ a sequence of two elements, we have the tensor product of

$$C^\bullet(x;R): \quad 0 \longrightarrow R \xrightarrow{r \mapsto \frac{r}{1}} R_x \longrightarrow 0$$

and

$$C^\bullet(y;R): \quad 0 \longrightarrow R \xrightarrow{r \mapsto \frac{r}{1}} R_y \longrightarrow 0\,,$$

which is

$$0 \longrightarrow R \otimes R \xrightarrow{\alpha} (R_x \otimes R) \oplus (R \otimes R_y) \xrightarrow{\beta} R_x \otimes R_y \longrightarrow 0\,.$$

The map α sends $1 \otimes 1$ to $(\frac{1}{1} \otimes 1, 1 \otimes \frac{1}{1})$. For β we have

$$\beta(\frac{1}{1} \otimes 1, 0) = (-1)\frac{1}{1} \otimes \frac{1}{1}\,, \quad \text{and} \quad \beta(0, 1 \otimes \frac{1}{1}) = \frac{1}{1} \otimes \frac{1}{1}\,.$$

Simplified, this becomes

$$C^\bullet(\{x,y\};R): \quad 0 \longrightarrow R \xrightarrow{1\mapsto(1,1)} R_x \oplus R_y \xrightarrow[(0,1)\ \mapsto\ \ 1]{(1,0)\ \mapsto\ -1} R_{xy} \longrightarrow 0\,.$$

Let's try to compute the cohomology $H^j(\{x,y\};R)$ for $j = 0, 2$. If $r \in R$ maps to zero in $R_x \oplus R_y$, so that $(\frac{r}{1}, \frac{r}{1}) = 0$, then there exist integers $a, b \geq 0$ such that $x^a r = y^b r = 0$. Equivalently, the ideal $(x,y)^c$ kills r for some $c \geq 0$. Thus

$$H^0(\{x,y\};R) \cong \bigcup_{c\geq 0} 0 :_R (x,y)^c$$

is the union of all annihilators of the ideals $(x,y)^c$. On the other hand, $H^2(\{x,y\};R) \cong R_{xy}/(R_x + R_y)$, which again is a less than completely satisfactory answer. Here is a more useful one:

Exercise 5.27. Observe that an element of $H^2(\{x,y\};R)$ can be written $\eta = \left[\frac{r}{(xy)^c}\right]$, that is, as an equivalence class of fractions in R_{xy}. Then show that $\eta = 0$ iff there exists $d \geq 0$ such that

$$r(xy)^d \in (x^{c+d}, y^{c+d})\,.$$

Conclude that for $\{x,y\}$ a regular sequence, $\eta = \left[\frac{r}{(xy)^c}\right]$ represents the zero element if and only if $r \in (x,y)^c$. State and prove the analogous statements for $H^j(\{x_1,\ldots,x_n\};R)$, $J \leq n$.

Remark 5.28. Unlike the Koszul complex on a sequence of elements, in which all the modules are free, the Čech complex is made up of direct sums of localizations of R. Specifically, we can see that $C^0(\underline{x}) \cong R$, while $C^1(\underline{x}) \cong R_{x_1} \oplus \cdots \oplus R_{x_n}$, and in general $C^k(\underline{x})$ is the direct sum of all localizations $R_{x_{i_1}\cdots x_{i_k}}$, where $1 \leq i_1 < \cdots < i_k \leq n$. In particular, $C^n(\underline{x}) \cong R_{x_1\cdots x_n}$. Note that in general $C^k(\underline{x})$ is not finitely generated over R, but that it is *flat*.

In this lecture we consider the top rung of the celebrated "hierarchy of rings": regular local rings. More generally, we will examine finitely generated modules of *finite projective dimension* over (local) rings. Regularity is characterized by finiteness of the projective dimension for every finitely generated module. Along the way, we will need to reconsider regular sequences, and prove the Auslander-Buchsbaum Theorem, which relates the existence of regular sequences to the finiteness of projective dimension.

In this lecture, we generally are concerned with a *local* ring $(R, \mathfrak{m}, \mathbb{K})$. This means that R is a Noetherian commutative ring with unique maximal ideal $\mathfrak{m}$, and $\mathbb{K} = R/\mathfrak{m}$.

7.1. Recall from Lecture 3 that the projective dimension $\operatorname{pd}_R M$ of a module M over a commutative ring R is by definition the minimal length of an R-projective resolution of M. Our first task is to give a homological characterization of projective dimension, at least in the case of a finitely generated module over a local ring. As we know, finitely generated projective modules over local rings are free, so a projective resolution has the form

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0 \tag{7.1.1}$$

with each F_i free of finite rank. In this case we also call (7.1.1) a *free resolution*. Note that by choosing bases for each F_i, we can write each φ_i as a matrix with entries from R.

Definition 7.2. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. A free resolution (7.1.1) is *minimal* if for each i, $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Equivalently, the entries of matrices representing the maps φ_i are all contained in $\mathfrak{m}$.

Proposition 7.3. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated nonzero R-module. Then $\operatorname{pd}_R M$ is the length of every minimal free resolution of M. Specifically, that value is given by*

$$\operatorname{pd}_R M = \inf\{i \geq 0 \mid \operatorname{Tor}_{i+1}^R(\mathbb{K}, M) = 0\}.$$

Proof. Let

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0 \tag{7.3.1}$$

be a minimal free resolution of M. Let β_i be the rank of the i^{th} free module F_i. We claim that $\beta_i = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$. To see this, apply $R/\mathfrak{m} \otimes_R -$ to (7.3.1), and consider the truncation at $F_0/\mathfrak{m}F_0$. The maps φ_i then give homomorphisms between free modules over $\mathbb{K}$. Note that $F_i/\mathfrak{m}F_i \cong \mathbb{K}^{\beta_i}$. Since (7.3.1) was chosen minimal, the entries of φ_i were in $\mathfrak{m}$, and as in Example 3.14 the induced map $\overline{\varphi_i} : F_i/\mathfrak{m}F_i \longrightarrow F_{i-1}/\mathfrak{m}F_{i-1}$ is the zero map. Now, $\operatorname{Tor}_i^R(\mathbb{K}, M)$ is the homology in the i^{th} position of the complex with zero differentials

$$\cdots \xrightarrow{0} \mathbb{K}^{\beta_n} \xrightarrow{0} \mathbb{K}^{\beta_{n-1}} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{K}^{\beta_1} \xrightarrow{0} \mathbb{K}^{\beta_0} \longrightarrow 0,$$

which is $\mathbb{K}^{\beta_i}$, as claimed.

Now it is clear that $\operatorname{pd}_R M$ is the least i such that $\beta_{i+1} = 0$ in some free resolution, which is the least i such that $\beta_{i+1} = 0$ in every minimal free resolution. $\square$

Definition 7.4. The numbers $\beta_i = \beta_i^R(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$ appearing in the proof of Proposition 7.3 are the *Betti numbers of M over R.*

Corollary 7.5. *The* global dimension *of a local ring $(R, \mathfrak{m}, \mathbb{K})$ is $\operatorname{pd}_R \mathbb{K}$. In particular, $\operatorname{pd}_R \mathbb{K} < \infty$ if and only if $\operatorname{pd}_R M < \infty$ for every finitely generated R-module M, and in this case $\operatorname{pd}_R M \leq \operatorname{pd}_R \mathbb{K}$.*

Proof. For any R-module M, we can compute $\operatorname{Tor}_i^R(\mathbb{K}, M)$ from a free resolution of $\mathbb{K}$. If $\operatorname{pd}_R \mathbb{K} < \infty$ then $\operatorname{Tor}_i^R(\mathbb{K}, M) = 0$ for $i > \operatorname{pd}_R \mathbb{K}$, and it follows that $\operatorname{pd}_R M \leq \operatorname{pd}_R \mathbb{K}$. □

The next definition is historically correct, but seems out of sequence here. Luckily, we will shortly prove that this is exactly the right place for it.

Definition 7.6. A local ring $(R, \mathfrak{m}, \mathbb{K})$ is *regular* if $\mathfrak{m}$ can be generated by $\dim R$ elements.

Remark 7.7. Recall that the minimal number of generators of $\mathfrak{m}$ is called the *embedding dimension of R*. It follows from Krull's (Generalized) Principal Ideal Theorem that $\mu(\mathfrak{m}) \geq \operatorname{height} \mathfrak{m} = \dim R$; regular rings are those for which equality obtains.

Geometrically, over a field of characteristic zero, say, regular local rings correspond to smooth (or "nonsingular") points on algebraic varieties. They are those for which the *tangent space* to the variety (at the specified point) has dimension no greater than that of the variety itself. It also turns out that $(R, \mathfrak{m})$ is a regular local ring if and only if the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ is a polynomial ring over the field $R/\mathfrak{m}$. We won't use this fact here, but it lends credence to the idea that all regular rings look more or less like polynomial rings over a field.

It will turn out below that a minimal generating set for the maximal ideal of a regular local ring is a regular sequence (as defined in Lecture 5). The regularity of this sequence is the key to our homological characterization of regular rings. Let us therefore consider regular sequences more carefully on their own terms.

Regular sequences and depth redux

The basic question we must address is how to establish the existence of a regular sequence, short of actually specifying elements. Specifically, given a ring R, an ideal $\mathfrak{a}$, and a module M, how can we tell whether $\mathfrak{a}$ contains an M-regular element? More generally, can we get a lower bound on $\operatorname{depth}_R(\mathfrak{a}, M)$?

Lemma 7.8. *Let R be a Noetherian ring and M, N finitely generated R-modules. Set $\mathfrak{a} = \operatorname{ann} M$. Then $\mathfrak{a}$ contains an N-regular element if and only if $\operatorname{Hom}_R(M, N) = 0$.*

Proof. ($\Longrightarrow$) We leave this direction as an easy exercise.
($\Longleftarrow$) Assume that $\mathfrak{a}$ consists entirely of zerodivisors on N. Then by Lemma 5.14, $\mathfrak{a}$ is contained in the union of the associated primes of N and, using prime avoidance, we can find $\mathfrak{p} \in \operatorname{Ass} N$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Localize at $\mathfrak{p}$ and reset notation to assume that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m} \in \operatorname{Ass} N$. (Since $\operatorname{Hom}_R(M, N)_{\mathfrak{p}} = \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, it suffices to show that the localized module is nonzero.) Then we have a surjection $M \longrightarrow M/\mathfrak{m}M \longrightarrow R/\mathfrak{m}$, and a monomorphism $R/\mathfrak{m} \hookrightarrow N$. The composition gives a nonzero homomorphism $M \longrightarrow N$. □

In combination with the following easy consequence of the long exact sequence of Ext, Lemma 7.8 will allow us to compute depths.

Proposition 7.9. *Let R be a ring and M, N R-modules. Suppose that there is an N-regular sequence $\underline{x} = x_1, \dots, x_n$ in $\mathfrak{a} := \operatorname{ann} N$. Then*

$$\operatorname{Ext}^n_R(M, N) \cong \operatorname{Hom}_R(M, N/\underline{x}N) \cong \operatorname{Hom}_{R/(\underline{x})}(M, N/\underline{x}N)\,.$$

Definition 7.10. We will say that a sequence of elements $x_1, \dots, x_n$ in an ideal $\mathfrak{a}$ of R is a *maximal regular sequence* on a module M (or *maximal M-regular sequence*) if $x_1, \dots, x_n$ is regular on M, and $x_1, \dots, x_n, y$ is not a regular sequence for any $y \in \mathfrak{a}$.

It's an easy exercise to show that in a Noetherian ring, every regular sequence can be lengthened to a maximal one. What's less obvious is that every regular sequence can be extended to one of the maximum possible length.

Theorem 7.11 (Rees)**.** *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}M \neq M$. Then any two maximal M-regular sequences in $\mathfrak{a}$ have the same length, namely*

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}\,.$$

One special case arises so often that we single it out. When $(R, \mathfrak{m}, \mathbb{K})$ is a local ring, we write simply $\operatorname{depth} M$ for $\operatorname{depth}_R(\mathfrak{m}, M)$.

Corollary 7.12. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. Then*

$$\operatorname{depth} M = \min\{i \mid \operatorname{Ext}^i_R(\mathbb{K}, M) \neq 0\}\,.$$

This fortuitous coincidence of an "elementary" property with a homological one accounts for the great power of the concept of depth. We note three immediate consequences; the first follows from the long exact sequence of Ext, and the second from computation of Ext via a projective resolution of the first argument.

Corollary 7.13 (The Depth Lemma)**.** *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of finitely generated modules over a ring R. Then for any ideal $\mathfrak{a}$,*

(1) $\operatorname{depth}_R(\mathfrak{a}, M) \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M'), \operatorname{depth}_R(\mathfrak{a}, M'')\}$,
(2) $\operatorname{depth}_R(\mathfrak{a}, M') \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M), \operatorname{depth}_R(\mathfrak{a}, M'') + 1\}$, *and*
(3) $\operatorname{depth}_R(\mathfrak{a}, M'') \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M') - 1, \operatorname{depth}_R(\mathfrak{a}, M)\}$.

Corollary 7.14. *For nonzero finitely generated M,* $\operatorname{depth}_R(\mathfrak{a}, M) \leq \operatorname{pd}_R R/\mathfrak{a}$.

Ideals so that equality is attained in Corollary 7.14, $\operatorname{depth}_R(\mathfrak{a}, R) = \operatorname{pd}_R R/\mathfrak{a}$, are called *perfect*. Our next main result is a substantial sharpening of this Corollary in a special case, the Auslander-Buchsbaum formula.

Theorem 7.15 (Auslander-Buchsbaum)**.** *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a nonzero finitely generated R-module of finite projective dimension. Then*

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R\,.$$

Proof. If $\operatorname{pd}_R M = 0$, then M is free, and $\operatorname{depth} M = \operatorname{depth} R$. We may therefore assume that $h := \operatorname{pd}_R M \geq 1$. If $h = 1$, let

$$0 \longrightarrow R^n \xrightarrow{\ \varphi\ } R^m \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M. We consider φ as an $m \times n$ matrix over R, with entries in $\mathfrak{m}$ by minimality. Apply $\operatorname{Hom}_R(\mathbb{K}, -)$ to obtain a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, R^n) \xrightarrow{\varphi_*} \operatorname{Ext}^i_R(\mathbb{K}, R^m) \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, M) \longrightarrow \cdots.$$

The entries of φ_* are the same as those of φ, after the identification $\operatorname{Ext}^i_R(\mathbb{K}, R^n) \cong \operatorname{Ext}^i_R(\mathbb{K}, R)^n$. Since each $\operatorname{Ext}^i_R(\mathbb{K}, R^n)$ is a vector space over $\mathbb{K}$, φ_* is identically zero. For each i, then, we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, R)^m \longrightarrow \operatorname{Ext}^i_R(\mathbb{K}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(\mathbb{K}, R)^n \longrightarrow 0.$$

It follows that $\operatorname{depth} M = \operatorname{depth} R - 1$, and we are done in this case.

If $h > 1$, take any exact sequence $0 \longrightarrow M' \longrightarrow R^m \longrightarrow M \longrightarrow 0$. Then $\operatorname{pd}_R M' = \operatorname{pd}_R M - 1$. By induction, $\operatorname{depth} M' = \operatorname{depth} R - h + 1$. But by the Depth Lemma, $\operatorname{depth} M' = \operatorname{depth} M + 1$, and the result follows. □

Corollary 7.16. *Over a local ring R, a module M of finite projective dimension has $\operatorname{pd}_R M \leq \operatorname{depth} R$.*

Remark 7.17. For noncommutative rings, this result is quite false. In fact, it is an open question in the theory of noncommutative Artin rings (the so-called *finitistic dimension conjecture*) whether the number

$$\operatorname{fin.dim.} R = \sup\{\operatorname{pd}_R M \mid \operatorname{pd}_R M < \infty\}$$

is finite.

Here are two amusing applications of the Auslander-Buchsbaum formula. You may need to look ahead to Lectures 9 and 10 for the relevant definitions.

Exercise 7.18. Let S be a regular local ring and $\mathfrak{a}$ an ideal such that $R = S/\mathfrak{a}$ is a Cohen-Macaulay ring with $\dim(R) = \dim(S) - 1$. Prove that $\mathfrak{a}$ is principal.

Exercise 7.19. Let S be a regular local ring and $\mathfrak{a}$ an ideal such that $R = S/\mathfrak{a}$ is a Gorenstein ring with $\dim(R) = \dim(S) - 2$. Prove that $\mathfrak{a}$ is a complete intersection (*i.e.,* generated by two elements).

Let us return to the singular world of regular rings. Here are two false statements that are nonetheless useful: every regular ring looks like a polynomial ring over a field, and regular sequences behave like polynomial indeterminates. Our next goal is to revise these statements so that they make sense, and then to prove them. To get an idea where we're headed, observe the following: For $R = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring, the sequence $\underline{x} = x_1, \ldots, x_n$ is a regular sequence. (This falls out immediately upon induction on n.) In particular, the Koszul complex on $\underline{x}$ is exact, so that $R/(\underline{x}) = \mathbb{K}$ has finite projective dimension. From Corollary 7.5 it follows that R has finite global dimension. All we need do is to replace $\mathbb{K}[x_1, \ldots, x_n]$ by an arbitrary regular local ring.

Here is a first easy lemma, the proof of which we leave as an exercise.

Lemma 7.20. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. If $\mathfrak{a}$ contains a regular sequence of length n, then $\mathfrak{a}$ has height at least n.*

Caution: the converse is quite false! (But see Lecture 9.)

Lemma 7.21. *Let $(R, \mathfrak{m})$ be a local ring and x a minimal generator of $\mathfrak{m}$ (so that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$). Then R is a regular local ring if and only if $R/(x)$ is so.*

Proof. Extend x to a full system of parameters $x = x_1, x_2, \ldots, x_n$. Then the maximal ideal of $R/(x)$ is generated by the images of $x_2, \ldots, x_n$, and has dimension one less than that of R. □

Lemma 7.22. *A regular local ring is a domain.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. The case $d = 0$ being trivial, assume first that $d = 1$. Then $\mathfrak{m}$ is a principal ideal, generated by some element $x \in R$. As $\dim R > 0$, $\mathfrak{m}$ is not nilpotent, but by Krull's Intersection Theorem, the intersection $\bigcap_{j \geq 0} x^j R$ is trivial. It follows that any element $a \in R$ can be written uniquely as a product of a unit times a power of x. If, then, $a = ux^p$ and $b = vx^q$ are such that $ab = 0$, with u and v units, we have $uvx^{p+q} = 0$, so that $x^{p+q} = 0$, a contradiction.

In the general case $d \geq 2$, use prime avoidance to find an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ outside the minimal primes of R. By induction and Lemma 7.21, $R/(x)$ is a domain, so (x) is a prime ideal. Now apply the same argument as above. □

Proposition 7.23. *Let $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements of a local ring $(R, \mathfrak{m})$. Consider the statements*

(1) $\underline{x}$ is an R-regular sequence.
(2) $\operatorname{height}(x_1, \ldots, x_i) = i$ for $i = 1, \ldots, n$.
(3) $\operatorname{height}(x_1, \ldots, x_n) = n$.
(4) $\underline{x}$ is part of a system of parameters for R.

Then (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). If R is a regular local ring, then each implication is an equivalence.

In fact, one can get by with much less than regularity; the last statement of the Lemma remains true if R is only *Cohen-Macaulay*. See Lecture 9.

Proof. (1) $\implies$ (2). By the definition of a regular sequence and Lemma 5.14, we have $\operatorname{height}(x_1) < \operatorname{height}(x_1, x_2) < \cdots$; now use Lemma 7.20.

(2) $\implies$ (3). This one is obvious.

(3) $\implies$ (4). If R has dimension n, we are done. If $\dim R > n$, then $\mathfrak{m}$ is not a minimal prime of $(\underline{x})$. It follows that there exists $x_{n+1} \in \mathfrak{m} \setminus (\underline{x})$ so that $\operatorname{height}(x_1, \ldots, x_n, x_{n+1}) = n + 1$. Continuing in this way, we obtain a system of parameters for R, as desired.

Now assume that R is regular. Take a system of parameters $\underline{x}$ such that $\underline{x}$ generate $\mathfrak{m}$. In particular, each x_i is a minimal generator of $\mathfrak{m}$. As R is a domain by Lemma 7.22, x_1 is certainly a nonzerodivisor. As $R/(x_1)$ is again a regular local ring by Lemma 7.21, we are done by induction. □

Putting the pieces together, we have shown that if $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m}$ is generated by a regular sequence, so $R/\mathfrak{m}$ has finite projective dimension. This leads us to the celebrated theorem of Serre.

Theorem 7.24 (Serre)**.** *The following are equivalent for a local ring $(R, \mathfrak{m})$.*

(1) R is regular.
(2) The global dimension of R is equal to $\dim R$.
(3) R has finite global dimension.

Proof. We have already established (1) $\implies$ (2), and (2) $\implies$ (3) is clear. For (3) $\implies$ (1), we go again by induction, this time on t, the minimal number of generators of $\mathfrak{m}$. If $t = 0$, then the zero ideal is maximal in R, so R is a field. Assume then that $t \geq 1$ and R has global dimension $g < \infty$. We first note that $\mathfrak{m} \notin \mathrm{Ass}(R)$: the finite free resolution of $R/\mathfrak{m}$ has all its matrices taking entries from $\mathfrak{m}$, so the final nonzero free module F_n is contained in $\mathfrak{m}F_{n-1}$. If $\mathfrak{m} \in \mathrm{Ass}(R)$, then $\mathfrak{m}$ is the annihilator of an element $a \in R$, so that $aF_n = 0$, a contradiction. By prime avoidance, then, we may take an element $x \in \mathfrak{m}$ outside of $\mathfrak{m}^2$ and the associated primes of R. The long exact sequence of Ext shows that $R/(x)$ has global dimension $g-1$, so $R/(x)$ is regular by induction. Finally, Lemma 7.21 implies that R is regular as well. □

Remark 7.25. So far we have clung to the case of local rings. A little care, however, allows one to generalize everything in this lecture to the case of graded rings, homogeneous elements, and homogeneous resolutions. In particular, we have the following theorem, for which an argument could be made that it is the second[8] theorem of commutative algebra. See [24] for a proof due to Schreyer. In particular, the proof given there, like Hilbert's original proof, does indeed produce a *free* resolution rather than merely a projective one.

Theorem 7.26 (Hilbert Syzygy Theorem)**.** *Let $\mathbb{K}$ be a field. Then every finitely generated module over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ has a free resolution of length at most n. If M is graded (with respect to any grading on $\mathbb{K}[x_1, \ldots, x_n]$) then the resolution can be chosen to be graded as well.*

[8]Since Hilbert's proof of the Syzygy Theorem (1890) uses his Basis Theorem (1888), there is at least one older.

Lecture 12. Connections with sheaf cohomology (GL)

You may have noticed that in previous lectures we have attached two meanings to the phrase "Čech complex". In Lecture 2, a complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ was defined for any sheaf $\mathcal{F}$ on a topological space and any open cover $\mathfrak{U}$ of X. Later, in lecture 5, we defined a complex $C^\bullet(\underline{x}; R)$ for any commutative ring R and sequence of elements $\underline{x}$. In one sense, the goal of this lecture is to reconcile this apparent overload of meaning: the two Čech complexes are "really" the same thing, at least up to a shift. In particular, the Čech cohomology of a ring is an invariant of the scheme structure. Specifically, we will prove that for an R-module M,

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(\operatorname{Spec} R \setminus V(\mathfrak{a}), \widetilde{M})$$

for all $j \geq 1$. See Theorem 12.28 for the precise statement and notation.

Our goal will take us fairly far afield, through the dense thickets of scheme theory and sheaf cohomology, flasque resolutions and cohomology with supports. We'll only have time to point out the windows at the menagerie of topics in this area. A more careful inspection is well worth your time. In particular, nearly everything in this lecture is covered more thoroughly in [55] and more intuitively in [30]. See also the prologue and epilogue of [100].

We begin with some background on sheaves.

Sheaf Theory from definitions to cohomology

Sheaves are *global objects* that are completely determined by *local data*. In fact, one of the points of sheaf theory is that sheaves make it possible to speak sensibly of "local properties". Historically, the notion of a sheaf seems to go back to complex analysis at the end of the 19th century, under the guise of *analytic continuation*, and was developed further in Weyl's 1913 book [149]. The first rigorous definition is due to Leray, and the basic properties were worked out in the Cartan seminar in the 1940s and 50s. Sheaves were imported to algebraic geometry by Serre in 1955 [130], when he also established the basics of sheaf cohomology. Grothendieck, in his 1955 Kansas lectures, introduced presheaves and the categorical approach to cohomology.

Here is the definition.

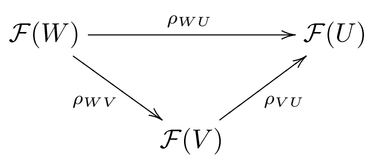
Definition 12.1. Let X be a topological space. A *sheaf* $\mathcal{F}$ of abelian groups on X consists of the data:

- an abelian group $\mathcal{F}(U)$ for every open set $U \subseteq X$, and
- a *restriction map* $\rho_{VU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ for each pair of open sets $V \subseteq U$, also sometimes written $|_V$,

subject to the following restrictions.

(0) (Sanity) $\mathcal{F}(\emptyset) = 0$ and $\rho_{UU} = \mathrm{id}_U$ for all open $U \subseteq X$.

(1) (Functoriality) If $W \subseteq V \subseteq U$ are open sets, then composition of restriction maps behaves well, *i.e.*, the diagram



commutes.

(2) (Locally zero implies zero) If $s, t \in \mathcal{F}(U)$ become equal after restriction to each V_α in an open cover $U = \bigcup_\alpha V_\alpha$, then they are equal.
(3) (Gluing) For any open cover $U = \bigcup_\alpha V_\alpha$, and any collection of elements $\{s_\alpha \in \mathcal{F}(V_\alpha)\}$ satisfying

$$s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$$

for all α, β, there exists $s \in \mathcal{F}(U)$ so that $s|_{V_\alpha} = s_\alpha$.

The first two axioms are fairly straightforward; they could be fancied up as, "$\mathcal{F}$ is a contravariant functor from the topology on X to abelian groups," or simplified as, "$\mathcal{F}$ is a sensible generalization of sending a set S to the continuous maps $S \longrightarrow \mathbb{C}$." The axiom of local zeroness and the gluing axiom codify the fact that sheaves are determined by their local data, and that local data varies "smoothly" over the space X. Note that (3) is an existence statement, while (2) asserts uniqueness. The two are sometimes combined into a single "sheaf axiom."

Before muddying the waters further with more definitions, let us have a concrete example which is also dear to our hearts.

Example 12.2. Let $X = \operatorname{Spec} R$ for some (Noetherian and commutative, like all rings in this lecture) ring R. Give X the Zariski topology (see Lecture 1), so that X has a base of open sets of the form

$$\begin{aligned} U_f &= \operatorname{Spec} R \setminus V(f) \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \text{ does not contain } f\} \end{aligned}$$

for each $f \in R$. Define a sheaf *of rings* on X, called the *structure sheaf* and denoted $\mathcal{O}_X$, by

$$\mathcal{O}_X(U_f) = R_f\,.$$

In other symbols, $\mathcal{O}_X$ assigns to every distinguished open set the ring of fractions $R[\frac{1}{f}]$. In particular, $\mathcal{O}_X(X) = R$. For two elements $f, g \in R$, we have

$$\begin{aligned} U_g \subseteq U_f &\iff V(f) \subseteq V(g) \\ &\iff f \mid g \\ &\iff \text{there is a natural localization map } R_f \longrightarrow R_g\,. \end{aligned}$$

Exercise 12.3. Check that $\mathcal{O}_X$ really is a sheaf. Specifically, check the axioms $(0)-(3)$ for the open sets U_f. Then show that any collection $\{\mathcal{F}(U_\alpha)\}$ satisfying the axioms and such that the $\{U_\alpha\}$ form a base for the topology of X defines a unique sheaf on X.

Example 12.4. If $\mathbb{K}$ is a field, then $\operatorname{Spec} \mathbb{K}$ is a single point, (0), which is both open and closed, and $\mathcal{O}_{\operatorname{Spec} \mathbb{K}}((0)) = \mathbb{K}$.

Example 12.5. Let $R = \mathbb{Z}$. Then $\operatorname{Spec} R$ consists of one point for each prime $p \in \mathbb{Z}$, plus a "generic" point corresponding to the zero ideal, whose closure is all of $\operatorname{Spec} \mathbb{Z}$. The distinguished open sets are the *cofinite* sets of primes, of the form

$$U_n = \{p \in \operatorname{Spec} \mathbb{Z} \mid p \nmid n\}$$

for $n \in \mathbb{Z} \setminus \{0\}$. For each nonzero n, we have $\mathcal{O}_{\mathbb{Z}}(U_n) = \mathbb{Z}[\frac{1}{n}]$.

Notation 12.6. We write $\Gamma(U, \mathcal{F})$ as a synonym[10] for $\mathcal{F}(U)$. The elements of the abelian group, or module, or ring, $\Gamma(U, \mathcal{F})$ are the *sections of $\mathcal{F}$ over U*. In particular, the elements of $\Gamma(X, \mathcal{F})$ are the *global sections of $\mathcal{F}$*.

Example 12.7. Let $X = \operatorname{Spec} R$. The sections of the structure sheaf $\mathcal{O}_X$ over a distinguished open set U_f are just fractions $\frac{r}{f^n}$ with $r \in R$ and $n \geq 0$. In particular, global sections of $\mathcal{O}_X$ are just elements of R.

A word of caution: while the sections of $\mathcal{O}_X$ over a distinguished open set are easy to identify from the definition, unexpected sections can crop up. In particular, not every section over an open set comes from restriction of a global section. Here are two examples.

Example 12.8. Let $X = \mathbb{C}$ with the usual complex topology, and let $\mathcal{F}$ be the sheaf of bounded holomorphic functions on X. Then the global sections of $\mathcal{F}$ are the bounded entire functions $\mathbb{C} \longrightarrow \mathbb{C}$. By Liouville's theorem, there are hardly any of these – only the constant functions qualify! For proper open sets, like discs, there are of course many nonconstant sections, which can't possibly be restrictions of global ones.

Example 12.9. Let $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$ (see Example 9.6), where $\mathbb{K}$ is some field. Set $X = \operatorname{Spec} R$ and let U be the open set $X \setminus \{(s^4, s^3t, st^3, t^4)\}$. As in Example 12.7, the global sections of $\mathcal{O}_X$ are just the elements of R. In particular, s^2t^2 is not a global section. The sections over U, however, include something new. To see this, we can use the surjection $\mathbb{K}[a, b, c, d] \longrightarrow R$ to think of X as an algebraic set embedded in $\mathbb{K}^4$. The open set U then corresponds to X with the origin deleted; equivalently, U is the subset of X where not all the coordinate functions a, b, c, d vanish simultaneously. If $a \neq 0$, then b^2/a represents $(s^3t)^2/s^4 = s^2t^2$, while if $d \neq 0$, then c^2/d represents s^2t^2. Since $a = d = 0$ forces $b = c = 0$, the two open sets defined by $a \neq 0$ and $b \neq 0$ cover all of U, and the sections b^2/a and c^2/d glue together to give $s^2t^2 \in \Gamma(U, \mathcal{F})$.

Being essentially categorical notions, sheaves of course come equipped with a notion of *morphisms*. These are the only reasonable thing: A morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of (group, module, ring...) homomorphisms $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ commuting with the restriction maps of $\mathcal{F}$ and $\mathcal{G}$.

Definition 12.10. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Define the kernel, image, and cokernel of φ by

(1) $\ker \varphi(U) = \ker(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;
(2) $\operatorname{image} \varphi(U) = \operatorname{image}(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;
(3) $\operatorname{coker} \varphi(U) = \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;

As benign as this definition is, it leads to serious difficulties very quickly.

Exercise 12.11.

(1) Check that $\ker \varphi$ is a sheaf.
(2) Show that $\operatorname{image} \varphi$ and $\operatorname{coker} \varphi$ satisfy the Sanity and Functoriality axioms to be sheaves.

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[10] This apparently unnecessary proliferation of symbols to represent the same thing is supposed to hint at the connections to come; see Lecture 6.

(3) Try (for a little while) to show that image φ and coker φ are sheaves.

This is where the smooth landscape of sheaf theory begins to show some wrinkles, not to say crevasses. Since it's the wellspring of much of what follows, we emphasize:

> The image and cokernel of a morphism of sheaves need not be a sheaf.

Here are two examples to indicate what goes wrong. One should be familiar from basic complex analysis, while one is from closer to home.

Example 12.12. Let $X = \mathbb{C}$ again, and define two sheaves over X: $\mathcal{F}(U)$ is the (additive) group of holomorphic functions $U \longrightarrow U$, and $\mathcal{G}(U)$ is the (multiplicative) group of nowhere vanishing holomorphic functions $U \longrightarrow U$. Define $\exp : \mathcal{F} \longrightarrow \mathcal{G}$ by

$$\exp(f)(z) = e^{2\pi i f(z)} .$$

Then the image of exp is not a sheaf. For any choice of a branch of the logarithm function, $f(z) = z$ is in the image of exp on the open subset of $\mathbb{C}$ defined by the branch. These open sets cover $\mathbb{C}$, but $f(z) = z$ has no global preimage on all of $\mathbb{C}$, so is not in the image of exp. In other words, $e^{2\pi i z}$ is locally invertible, but has no analytic global inverse.

Example 12.13. Let $(R, \mathfrak{m})$ be a local ring and let $\{x_1, \dots, x_n\}$ be a set of generators for $\mathfrak{m}$. Set $X = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, the punctured spectrum of R. Let $\mathcal{O}_U$ be the sheaf obtained by restricting the structure sheaf of Spec R to U. Define a morphism $\varphi : \mathcal{O}_U^n \longrightarrow \mathcal{O}_U$ by $\varphi(s_1, \dots, s_n) = \sum_i s_i x_i$. Then the image of φ is not a sheaf. To see this, put $U_i = U_{x_i}$ and note that $U_1, \dots, U_n$ cover U. On each U_i we have $\varphi(0, \dots, \frac{1}{x_i}, \dots, 0) = 1$. Thus we have an open cover of U and a section of image φ on each constituent so that the sections agree on the overlaps, but the sections cannot be glued together.

Routing around the failure of the category of sheaves to be closed under the operations of taking images and cokernels involves two definitions: *presheaves* and *sheafification*.

Definition 12.14. A collection $\{\mathcal{F}(U), \rho_{VU}\}$ satisfying the Sanity and Functoriality axioms of Definition 12.1 is a *presheaf*.

Example 12.15. Let $X = \{a, b\}$ be the two-point space with the discrete topology. Define $\mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = 0$ and $\mathcal{F}(X) = \mathbb{Z}$. Check that $\mathcal{F}$ is a presheaf (of abelian groups) and not a sheaf.

It follows from Example 12.11(b) that the image and cokernel of a morphism of sheaves are presheaves. Even if this weren't enough reason to consider them, we'll see that one can do cohomology with presheaves as well. First, we mention the procedure for obtaining a sheaf from a presheaf. This requires one preliminary definition, which is the counterpart in sheaf theory of the local notion of a *germ* of functions.

Definition 12.16. Let $x \in X$ and let $\mathcal{F}$ be a presheaf on X. The *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_{x,X} := \varinjlim \mathcal{F}(U) ,$$

where the direct limit is taken over the directed system of all open sets U containing x, partially ordered by inclusion.

Exercise 12.17. Check that for $\mathfrak{p} \in X = \operatorname{Spec} R$, the stalk of the structure sheaf $\mathcal{O}_X$ over $\mathfrak{p}$ is the local ring $R_{\mathfrak{p}}$. We say that $(X, \mathcal{O}_X)$ is a *locally ringed space.*

Now we define the sheafification of a presheaf.

Definition 12.18. Let $\mathcal{F}$ be a presheaf on X. The *sheafification* of $\mathcal{F}$ is the unique sheaf $\widetilde{\mathcal{F}}$ and morphism of presheaves $\mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ so that the stalk $\mathcal{F}_{x,X} \longrightarrow \widetilde{\mathcal{F}}_{x,X}$ is an isomorphism for all $x \in X$.

We remark that sheafifications always exist ([55, II.1.2]).
We can now remedy our embarrassing lack of images and cokernels.

Definition 12.19. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of (pre)sheaves.

(1) The *image sheaf of* φ is the sheafification of the presheaf image φ.
(2) The *cokernel sheaf of* φ is the sheafification of the presheaf coker φ.
(3) We say that φ is *surjective* if the image sheaf of φ is equal to $\mathcal{G}$.
(4) A sequence of morphisms of sheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is *exact at* $\mathcal{G}$ if $\ker \psi$ is equal to the image sheaf of φ.

Lemma 12.20. *A morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ over X is surjective (bijective) if and only if the stalk $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is so for each $x \in X$. In particular, a sequence $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ is exact if and only if the sequence of stalks $\mathcal{F}_{x,X} \longrightarrow \mathcal{G}_{x,X} \longrightarrow \mathcal{H}_{x,X}$ is exact for every $x \in X$.*

Exercise 12.21. Check that in Exercise 12.13 above, φ is surjective. Note that a surjective sheaf morphism need not restrict to surjective maps over open sets! In each case, however, φ restricts to a surjective map over "small enough" open sets. What about Exercise 12.12?

We also define the important notion of *sheaves associated to modules*.

Definition 12.22. Let R be a ring and $X = \operatorname{Spec} R$. For an R-module M we define *the sheafification of* M to be the unique sheaf $\widetilde{M}$ with $\widetilde{M}(U_f) \cong M_f$ for all $f \in R$. A sheaf obtained by sheafifying an R-module is called *quasi-coherent*; if the module is finitely generated over R, then the sheaf is *coherent*.

Note that this definition must be modified slightly in the graded case; see Lecture 13.

The sheafification of an R-module actually gives rise to a *sheaf of $\mathcal{O}_X$-modules*, that is, $\Gamma(U, \widetilde{M})$ is a $\Gamma(U, \mathcal{O}_X)$-module for every U.

Sheafification of R-modules is an exact functor, since by Lemma 12.20 we can measure exactness on stalks. We do, however, lose some information. For example, our best candidate for a projective object is the sheafification of the free module R, that is, $\mathcal{O}_X$ itself. It turns out, though, that there are surjective maps to $\mathcal{O}_X$ that are not split! See the next lecture. In particular, it's not at all clear how to take a projective resolution of an $\mathcal{O}_X$-module. We'll deal with this shortly.

First, let's investigate the functor that goes in the opposite direction: "take global sections."

Proposition 12.23. *The global sections functor $\Gamma(X, -)$ is left-exact. Specifically, if*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence of sheaves, then

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H})$$

is an exact sequence of abelian groups.

This proposition points toward the solution of the problem of the missing projectives, and we finally reach the object of this lecture.

Definition 12.24. Let R be a ring and $\mathcal{F}$ a sheaf of modules over $X = \operatorname{Spec} R$ (i.e., an $\mathcal{O}_X$-module). Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^1} \mathcal{I}^1 \xrightarrow{d^2} \dots$$

be an *injective* resolution of $\mathcal{F}$. Apply the global sections functor to the truncation of this resolution to obtain

$$0 \longrightarrow \Gamma(X, \mathcal{I}^0) \xrightarrow{\Gamma d^1} \Gamma(X, \mathcal{I}^1) \xrightarrow{\Gamma d^2} \dots ,$$

a complex of R-modules. Then the j^{th} *sheaf cohomology of* $\mathcal{F}$ is then

$$H^j(X, \mathcal{F}) = \ker \Gamma d^{j+1} / \operatorname{image} \Gamma d^j .$$

In particular, we have $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

Remark 12.25. In order for this definition to make sense, we must be able to take injective resolutions of $\mathcal{O}_X$-modules. Given our hardships with projectives noted above, this is cause for trepidation. Luckily, the category of modules over any locally ringed space $(X, \mathcal{O}_X)$ has *enough injectives*. We can see this as follows: for any $\mathcal{O}_X$-module $\mathcal{F}$, the stalk $\mathcal{F}_x$ at a point $x \in X$ embeds in an injective module $\mathcal{E}_x$ over the local ring $\mathcal{O}_{X,x}$. Set $\mathcal{E} = \prod_x \mathcal{E}_x$. Then the natural composition $\mathcal{F} \longrightarrow \mathcal{E}$ is an embedding and $\mathcal{E}$ is an injective $\mathcal{O}_X$-module.

Exercise 12.26. Why doesn't Remark 12.25 work for projectives?

The definition of sheaf cohomology given here has two slight problems. It's essentially impossible to compute, and it lengthens our already lengthy list of cohomology theories to keep track of. We can solve both these problems at once.

12.27. Let R be a Noetherian ring and $\mathfrak{a} = (x_1, \dots, x_n)$ an ideal of R. Put $X = \operatorname{Spec} R$, $V(\mathfrak{a})$ the closed set of X defined by $\mathfrak{a}$, and $U = X \setminus V(\mathfrak{a})$. Let $\mathfrak{U} = \{U_i\}$ be the open cover of U given by $U_i = X \setminus V(x_i)$. Let M be an arbitrary R-module and $\widetilde{M}$ the sheafification. Then $\Gamma(U_i, \widetilde{M}) \cong M_{x_i}$.

We have two complexes associated to this data: the Čech complex $C^\bullet(\underline{x}; M)$ of the sequence $\underline{x} = x_1, \dots, x_n$ and the module M, which has the R-module

$$\bigoplus_{1 \le i_1, < \cdots < i_k \le n} M_{x_{i_1} x_{i_2} \cdots x_{i_k}}$$

in the k^{th} position, and the topological Čech complex $\check{C}^\bullet(\mathfrak{U}, \widetilde{M}|_U)$ associated to the open cover $\mathfrak{U}$ and sheaf $\widetilde{M}|_U$, which has

$$\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} \Gamma(U_{i_1} \cap \cdots \cap U_{i_{k+1}}, \widetilde{M}|_U)$$

in the k^{th} spot. Recall that $\Gamma(U, \widetilde{M}|_U)$ does not appear in the topological complex, but (see Exercise 2.8) is naturally isomorphic to the kernel of the zeroth differential.

Theorem 12.28. *In the situation of 12.27, we have an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0(X, \widetilde{M}) \longrightarrow H^0(U, \widetilde{M}|_U) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and isomorphisms for all $j \geq 1$

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U, \widetilde{M}|_U)$$

between the local cohomology of M *with support in* $\mathfrak{a}$ *and the sheaf cohomology of* $\widetilde{M}$ *over* U.

Remark 12.29. In the exact sequence of the Theorem, the inclusion of $H^0_{\mathfrak{a}}(M)$ into $H^0(X, \widetilde{M}) = \Gamma(X, \widetilde{M}) \cong M$ is the natural one; its elements correspond to sections s of $\widetilde{M}$ supported only on $V(\mathfrak{a})$, so that $s_{\mathfrak{p}} = 0$ unless $\mathfrak{p} \supseteq \mathfrak{a}$. Such a section dies when restricted to $U = X \setminus V(\mathfrak{a})$. The cokernel $H^1_I(M)$ measures the obstruction to extending a section of $\widetilde{M}$ over U to a global one. In particular, this has the following useful consequence.

Corollary 12.30. *Let* R *be a Noetherian ring,* M *a finitely generated* R*-module, and* $\mathfrak{a}$ *an ideal. If* $\mathfrak{a}$ *contains a regular sequence of length* 2 *on* M*, then every section of* $\widetilde{M}$ *over* $U = \operatorname{Spec} R \setminus V(\mathfrak{a})$ *extends to a global section.*

Remark 12.31. The functorial road to sheaves that we've followed above is the most common modern approach. In Godement's influential book [41], however, the order of exposition (open sets $\rightsquigarrow$ stalks) was the reverse. Suppose that $\mathcal{F}$ is a sheaf on some space X, and let E be the topological space with underlying set $\prod_{x \in X} \mathcal{F}_{x,X}$, the product of all stalks of $\mathcal{F}$. Topologize E by (1) defining a map $\pi : E \longrightarrow X$ sending each $\mathcal{F}_{x,X}$ to x and (2) insisting that each stalk have the discrete topology and π be continuous. One can then prove ([41][Théorème 1.2.1]) that the thus constructed *espace étalé* ("flattened") or *total sheaf space* E has the property that for every open U, $\mathcal{F}(U)$ is naturally identified with the set $C_0(U, E)$ of continuous maps $f : U \longrightarrow E$ such that $f(x) \in \mathcal{F}_{x,X}$. Thus every sheaf *is* a sheaf of functions, as in Lecture 2.

Flasque sheaves and cohomology with supports

It follows immediately from the definition that injective sheaves of $\mathcal{O}_X$-modules are *acyclic* for sheaf cohomology, *i.e.,* $H^j(X, \mathcal{I}) = 0$ for any $j > 0$ and any injective sheaf $\mathcal{I}$. In fact, injective objects in any category are acyclic for any right-derived functors of covariant functors, since we compute such things from injective resolutions. In some sense, injectivity is overkill for our purposes. As part of their extreme acyclicity, injective objects are also extremely complicated in general (for example, over a Noetherian ring R, indecomposable injective R-modules are in one-to-one correspondence with $\operatorname{Spec} R$ itself). For the particular application we have in mind, then, it's worth searching for another class of sheaves which are still acyclic for the global sections functor (that is, for sheaf cohomology), but perhaps more manageable in whatever sense we can manage. If we're lucky, this new class will also be more "intrinsic" to sheaf cohomology, rather than being a generic categorical notion, depending on the whole category of sheaves.

Definition 12.32. A sheaf $\mathcal{F}$ on a topological space X is *flasque* (also "flabby" or "scattered") if for every pair of open sets $V \subseteq U$ in X, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective. In particular, sections on open subsets always extend to sections on the whole space.

Here are three examples. The third, of course, is the point.

Example 12.33. Let X be connected and $\mathcal{F}$ a *constant* sheaf over X, so for each open set U, $\mathcal{F}(U) = A$ for some fixed abelian group A. Then $\mathcal{F}$ is flasque.

Example 12.34. *Skyscraper sheaves* are flasque. Fix $x \in X$ and some abelian group A, and define $F(U) = A$ if $x \in U$, $\mathcal{F}(U) = 0$ otherwise. Then every section of $\mathcal{F}$ extends to all of X.

Example 12.35. Injective sheaves are flasque. More specifically, if a sheaf $\mathcal{M}$ of modules over a locally ringed space $(X, \mathcal{O}_X)$ is an injective object in the category of $\mathcal{O}_X$-modules, then $\mathcal{M}$ is flasque. To see this, let $V \subseteq U$ be open sets in X, and let $\mathcal{O}_V$, $\mathcal{O}_U$ be the structure sheaf extended by 0 outside V and U respectively. Then $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ is an embedding of $\mathcal{O}_X$-modules, so $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{M})$ surjects onto $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{M})$. (We haven't talked about Hom of sheaves, but you can pretend it works just like for modules.) Now, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{M})$ is naturally isomorphic to $\Gamma(U, \mathcal{M})$ (check this!). It follows that $\Gamma(U, \mathcal{M}) \twoheadrightarrow \Gamma(V, \mathcal{M})$, and $\mathcal{M}$ is flasque.

Remark 12.36. Here are some further basic properties of flasque sheaves, most of which are easy, or can be taken for granted at a first pass, or both. We give only first-order approximations to the proofs.

(1) If, in the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0\,,$$

$\mathcal{F}$ is flasque, then

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$

is exact. Remember, Γ is left-exact in general, so all we need check is surjectivity. Note that $\mathcal{G} \longrightarrow \mathcal{H}$ is surjective on stalks, and use the definition of stalks as direct limits to get surjectivity for small neighborhoods. Zorn's Lemma provides a maximal extension from such neighborhoods, which by the flasque condition must be all of X.

(2) Quotients of flasque sheaves by flasque subsheaves are flasque. Use the surjectivity from above to extend a section of the quotient to a global section of the large sheaf.

(3) Direct limits of flasque sheaves are flasque over Noetherian topological spaces. The issue here is that $\varinjlim(\mathcal{F}_\alpha(U)) = (\varinjlim \mathcal{F}_\alpha)(U)$ when X is Noetherian. In particular, arbitrary direct sums of flasque sheaves are flasque.

(4) The sheafification of an injective module over a Noetherian ring R is a flasque sheaf. See [55, III.3.4].

Example 12.37. Let $X = \mathbb{P}^1 = \mathrm{Proj}\, \mathbb{K}[x, y]$ be the projective line over an algebraically closed field (e.g., the Riemann sphere $\mathbb{C} \cup \{\infty\}$). Let $\mathcal{O}_X$ be the structure sheaf of X, and $\mathcal{K}$ the constant sheaf associated to the function field $\widetilde{\mathbb{K}}$ of X. Then $\mathcal{O}_X$ embeds naturally in $\mathcal{K}$. The quotient sheaf $\mathcal{K}/\mathcal{O}_X$ can be thought of (stalkwise) as the direct sum over all $x \in X$ of the skyscraper sheaf $\mathcal{K}/\mathcal{O}_{X,x}$ at x, so is flasque. Thus

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O}_X \longrightarrow 0$$

is a *flasque resolution* of $\mathcal{O}_X$. If we take global sections, we get an exact sequence

$$0 \longrightarrow \mathbb{K}[x, y] \longrightarrow \widetilde{\mathbb{K}} \longrightarrow \widetilde{\mathbb{K}}/\mathbb{K}[x, y] \longrightarrow 0\,.$$

As soon as we prove that sheaf cohomology can be computed via flasque resolutions, this will show that $H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for all $j \geq 1$. This example can be souped up beyond recognition and has connections with the Residue Theorem ([55, p. 248]).

If you're willing to accept the assertions above, then you must accept

Proposition 12.38. *Flasque sheaves over a Noetherian locally ringed space $(X, \mathcal{O}_X)$ are acyclic for the global sections functor, i.e., $H^j(X, \mathcal{F}) = 0$ for all $j \geq 1$ if $\mathcal{F}$ is flasque. In particular, resolutions by flasque sheaves may be used to compute sheaf cohomology in place of injective resolutions.*

Proof. Let $\mathcal{F}$ be flasque, and embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. The quotient $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ is also flasque. Taking global sections gives an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{Q}) \longrightarrow 0\,.$$

The long exact sequence of cohomology thus gives that $H^1(X, \mathcal{F}) = 0$ and $H^j(X, \mathcal{F}) \cong H^{j-1}(X, \mathcal{Q})$ for all $j > 1$. By induction on j, $H^j(X, \mathcal{F}) = 0$ for all $j \geq 1$. □

Remark 12.39. Weirdly enough, sheaf cohomology is trivial for quasi-coherent sheaves over Noetherian affine schemes, that is, spaces of the form $\operatorname{Spec} R$ for Noetherian commutative rings R. Perhaps we should have mentioned this earlier, since this situation would appear to be one of our main motivations. To be specific, let R be such a ring, and let M be an arbitrary R-module. Then an injective resolution of M sheafifies to a flasque resolution of $\widetilde{M}$ (since sheafification is exact), which we can use to compute $H^j(X, \widetilde{M})$. Applying Γ, though, just gets us back the original injective resolution of M! This is exact by design, so $H^j(X, \widetilde{M}) = 0$. In fact, the converse is true as well.

Theorem 12.40 (Serre [132])**.** *Let X be a Noetherian scheme. Then the following are equivalent.*

- *(1) X is affine;*
- *(2) $H^k(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ on X and all $j > 0$.*

Consider again the exact sequence and isomorphisms of Theorem 12.28:

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow H^0(U, \widetilde{M}|_U) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0\,,$$

and $H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U, \widetilde{M}|_U$ for $j \geq 1$. One way of looking at this theorem is that sheaf cohomology of $\widetilde{M}$ on $\operatorname{Spec} R$ away from $V(\mathfrak{a})$ controls the local cohomology of M with support in $\mathfrak{a}$. In other words, only the *support* of $\mathfrak{a}$ matters. Following this idea leads naturally to *cohomology with supports*.

Definition 12.41. Let Z be a close set in some topological space X. For any sheaf $\mathcal{F}$ over X, the group of *sections of $\mathcal{F}$ with support in Z* is the kernel of the restriction map from $\Gamma(X, \mathcal{F})$ to $\Gamma(X \setminus Z, \mathcal{F})$. That is,

$$\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \setminus Z, \mathcal{F}))\,.$$

As with global sections, the functor "take sections with support in Z" is left-exact. (Check this! The Snake Lemma should come in handy.)

Definition 12.42. The j^{th} *local cohomology of $\mathcal{F}$ with support in Z* is

$$H^j_Z(X, \mathcal{F}) := R^j\Gamma_Z(X, \mathcal{F})\,.$$

In other words, to compute the cohomology of $\mathcal{F}$ with support in Z, take an injective resolution of $\mathcal{F}$, apply $\Gamma_Z = H^0_Z$, and take cohomology.

Remark 12.43. Let X, Z, and $\mathcal{F}$ be as in the definition above, and put $U = X \setminus Z$. Then there is a natural exact sequence

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}) \tag{12.43.1}$$

given by the definition of Γ_Z. Assume for the moment that $\mathcal{F}$ is flasque. Then every section of $\mathcal{F}$ over U extends to a global section, so (12.43.1) can be extended to a full short exact sequence.

It follows by taking flasque resolutions that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$0 \longrightarrow H^0_Z(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}|_U) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^j(X, \mathcal{F}) \longrightarrow H^j(U, \mathcal{F}|_U) \longrightarrow H^{j+1}_Z(X, \mathcal{F}) \longrightarrow \cdots$$

Together with Theorem 12.40 and the Snake Lemma, this gives our best connection with local cohomology.

Theorem 12.44. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and M an R-module. Set $X = \operatorname{Spec} R$, $Z = V(\mathfrak{a})$, and $U = X \setminus Z$. Then for each $j \geq 1$ we have*

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U, \widetilde{M}|_U) \cong H^j_Z(X, \widetilde{M})\,.$$

References

[1] A. Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), 269–290. 241, 242

[2] E. Artin, *Galois theory*, Notre Dame Mathematical Lectures **2**, University of Notre Dame Press, South Bend, Indiana, 1959. 83

[3] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ontario, 1969. 2, 4, 5, 6, 40, 78, 82, 250

[4] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. 94

[5] H. Bass, *Some problems in "classical" algebraic K-theory*, in: Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Seattle, 1972), 3–73, Lecture Notes in Math. **342**, Springer, Berlin, 1973. 31

[6] I. N . Bernšteĭn, *Analytic continuation of generalized functions with respect to a parameter*, Funkcional. Anal. i Priložen. **6** (1972), 26–40. 164

[7] J.-E. Björk, *Rings of differential operators*, North-Holland Mathematical Library **21**, North-Holland Publishing Co., Amsterdam, 1979. 161, 164, 165, 233

[8] J.-E. Björk, *Analytic $\mathcal{D}$-modules and applications*, Mathematics and its Applications **247**, Kluwer Academic Publishers Group, Dordrecht, 1993. 13, 234

[9] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics **82**, Springer-Verlag, New York, 1982. 13, 181, 182

[10] M. Brodmann, *Einige Ergebnisse aus der lokalen Kohomologietheorie und ihre Anwendung*, Osnabrücker Schriften zur Mathematik, Reihe M: Mathematische Manuskripte **5**, Universität Osnabrück, Fachbereich Mathematik, Osnabrück, 1983. 142

[11] M. P. Brodmann and A. Lashgari Faghani, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (2000), 2851–2853. 228

[12] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics **60**, Cambridge University Press, Cambridge, 1998. 133, 138

[13] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998. 78, 106, 107, 119, 137, 157, 203, 250

[14] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer-Verlag, Berlin, 1988. 93

[15] J. Carlson, S. Müller-Stach, and C. Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics **85**, Cambridge University Press, Cambridge, 2003. 190

[16] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, New Jersey, 1956. 182, 191

[17] E. Cattani, C. D'Andrea, and A. Dickenstein, *The A-hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), 179–207. 243

[18] E. Cattani, A. Dickenstein, and B. Sturmfels, *Rational hypergeometric functions*, Compositio Math. **128** (2001), 217–239. 241

[19] CoCoATeam, CoCoA*: a system for doing Computations in Commutative Algebra*, available at `http://cocoa.dima.unige.it`. 40

[20] S. C. Coutinho, *A primer of algebraic D-modules*, London Mathematical Society Student Texts **33**, Cambridge University Press, Cambridge, 1995. 160, 164

[21] R. C. Cowsik and M. V. Nori, *Affine curves in characteristic p are set theoretic complete intersections*, Invent. Math. **45** (1978), 111–114. 80, 115

[22] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Amer. Math. Soc., Providence, RI, 1999. 241

[23] D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1997. 40, 48

[24] D. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, Graduate Texts in Mathematics **185**, Springer, New York, 2005. 73

[25] M. Deuring, *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*, Abh. Math. Sem. Hansischen Univ. **14** (1941), 197–272. 225

[26] A. Dimca, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004. 182

[27] W. Dwyer, J. P. C. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. Math., to appear.

[28] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995. 32, 53, 113

[29] D. Eisenbud, D. R. Grayson, M. Stillman, and B. Sturmfels (Eds.), *Computations in algebraic geometry with Macaulay 2*, Algorithms and Computation in Mathematics **8**, Springer-Verlag, Berlin, 2002. 40

[30] D. Eisenbud and J. Harris, *Schemes: the language of modern algebraic geometry*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992. 120

[31] N. D. Elkies, *The existence of infinitely many supersingular primes for every elliptic curve over* $\mathbf{Q}$, Invent. Math. **89** (1987), 561–567. 225

[32] E. G. Evans Jr. and P. A. Griffith, *Local cohomology modules for normal domains*, J. London Math. Soc. (2) **19** (1979), 277–284. 79

[33] G. Faltings, *Über lokale Kohomologiegruppen hoher Ordnung*, J. Reine Angew. Math. **313** (1980), 43–51.

[34] Y. Flix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics **205**, Springer-Verlag, New York, 2001.

[35] O. Forster, *Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring*, Math. Z. **84** (1964), 80–87. 113

[36] R. Fossum, H.-B. Foxby, P. Griffith, and I. Reiten, *Minimal injective resolutions with applications to dualizing modules and Gorenstein modules*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 193–215.

[37] H.-B. Foxby and S. Iyengar, *Depth and amplitude for unbounded complexes*, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math. **331** (2003), 119–137. 76, 77

[38] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), 14–19. 239, 241, 242

[39] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), 12–26. Correction in ibid, **27** (1993), 91. 241, 242

[40] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. 13

[41] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1973. 13, 22, 28, 126, 191

[42] S. Goto and K.-i. Watanabe, *On graded rings I*, J. Math. Soc. Japan **30** (1978), 179–213. 222

[43] V. E. Govorov, *On flat modules*, Sibirsk. Mat. Ž. **6** (1965), 300–304. 32

[44] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at `http://www.math.uiuc.edu/Macaulay2/`.

[45] G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 2.0, *A computer algebra system for polynomial computations*, Centre for Computer Algebra, University of Kaiserslautern, 2001, available at `http://www.singular.uni-kl.de`. 40

[46] G.-M. Greuel and G. Pfister, *A* **Singular** *introduction to commutative algebra*, Springer-Verlag, Berlin, 2002. 40, 43, 45

[47] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons Inc., New York, 1994. 13, 28

[48] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **29** (1966), 95–103. 190

[49] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, Séminaire de Géométrie Algébrique du Bois-Marie, North-Holland Publishing Co., 1968. 105, 226

[50] R. Hartshorne, *Local cohomology. A seminar given by A. Grothendieck*, Lecture Notes in Mathematics **41**, Springer-Verlag, Berlin-New York 1967.

[51] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics **20**, Springer–Verlag, Berlin, 1966. 207

[52] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450. 139, 142
[53] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1969/1970), 145–164. 207, 226
[54] R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 5–99. 258
[55] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York, 1977. 13, 120, 124, 127, 128, 190
[56] R. Hartshorne and R. Speiser, *Local cohomological dimension in characteristic p*, Ann. of Math. (2) **105** (1977), 45–79. 219, 222
[57] R. Hartshorne, *Complete intersections in characteristic* $p > 0$, Amer. J. Math. **101** (1979), 380–383. 100, 115
[58] R. C. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. (2) **156** (2002), 695–712. 214
[59] D. Helm and E. Miller, *Bass numbers of semigroup-graded local cohomology*, Pacific J. Math. **209** (2003), 41–66. 209
[60] J. Herzog, *Ringe der Charakteristik p und Frobeniusfunktoren*, Math. Z. **140** (1974), 67–78. 211
[61] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. (2) **96** (1972), 318–337. 83, 204
[62] M. Hochster, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conference Series in Mathematics **24**, American Mathematical Society, Providence, Rhode Island, 1975. 214
[63] M. Hochster, *Some applications of the Frobenius in characteristic* 0, Bull. Amer. Math. Soc. **84** (1978), 886–912. 83
[64] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. Algebra **84** (1983), 503–553. 214
[65] M. Hochster and J. A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058. 91
[66] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.
[67] M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. of Math. (2) **135** (1992), 53–89. 214
[68] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175. 92, 213
[69] M. Hochster and J. L. Roberts, *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), 117–172.
[70] R. Hotta, *Equivariant D-modules*, 1998. `arXiv:math.RT/9805021` 241
[71] C. Huneke, *Problems on local cohomology*, in: Free resolutions in commutative algebra and algebraic geometry (Sundance, Utah, 1990), 93–108, Res. Notes Math. **2**, Jones and Bartlett, Boston, Massachusetts, 1992. 226
[72] C. Huneke and G. Lyubeznik, *On the vanishing of local cohomology modules*, Invent. Math. **102** (1990), 73–93.
[73] C. L. Huneke and R. Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765–779. 226
[74] M.-N. Ishida, *Torus embeddings and dualizing complexes*, Tôhoku Math. J. (2) **32** (1980), 111–146. 207
[75] M.-N. Ishida, *The local cohomology groups of an affine semigroup ring*, in: Algebraic geometry and commutative algebra in Honor of Masayaoshi Nagata, vol. I, Kinokuniya, Tokyo, 1987, 141–153. 207
[76] B. Iversen, *Cohomology of sheaves*, Universitext, Springer-Verlag, Berlin, 1986. 13
[77] S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), 545–567.
[78] G. Kalai, *Many triangulated spheres*, Discrete Comput. Geom. **3** (1988), 1–14. 154
[79] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs **217**, American Mathematical Society, Providence, Rhode Island, 2003. 233
[80] M. Katzman, *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra **252** (2002), 161–166. 229

[81] K. Khashyarmanesh and Sh. Salarian, *On the associated primes of local cohomology modules*, Comm. Alg. **27** (1999), 6191–6198. 228
[82] L. Kronecker, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, J. Reine Angew. Math. **92** (1882), 1–121. 113
[83] E. Kunz, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784. 210
[84] E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013.
[85] E. Kunz, *Introduction to commutative algebra and algebraic geometry*, Birkhä user Boston, Inc., Boston, MA, 1985. 2, 3
[86] S. Lang and H. Trotter, *Frobenius distributions in* GL_2*-extensions*, Lecture Notes in Mathematics **504**, Springer-Verlag, Berlin-New York, 1976. 225
[87] D. Lazard, *Sur les modules plats*, C. R. Acad. Sci. Paris **258** (1964), 6313–6316. 32
[88] A. Leykin, *Constructibility of the set of polynomials with a fixed Bernstein-Sato polynomial: an algorithmic approach*, in: Effective methods in rings of differential operators, J. Symbolic Comput. **32** (2001), 663–675. 236
[89] A. Leykin, *Computing local cohomology in Macaulay 2*, in: Local cohomology and its applications (Guanajuato, 1999), 195–205, Lecture Notes in Pure and Appl. Math. **226**, Dekker, New York, 2002.
[90] A. Leykin, *D-modules for Macaulay 2*, in: Mathematical software (Beijing, 2002), 169–179, World Sci. Publishing, River Edge, New Jersey, 2002. 163
[91] A. Leykin, *Algorithmic proofs of two theorems of Stafford*, J. Symbolic Comput. **38** (2004), 1535–1550. 161
[92] H. Lindel, *On the Bass-Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. **65** (1981/82), 319–323. 31
[93] G. Lyubeznik, *A survey of problems and results on the number of defining equations*, in: Commutative algebra (Berkeley, CA, 1987), 375–390, Math. Sci. Res. Inst. Publ. **15**, Springer, New York, 1989. 113
[94] G. Lyubeznik, *The number of defining equations of affine algebraic sets*, Amer. J. Math. **114** (1992), 413–463. 115
[95] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), 41–55. 227, 228
[96] G. Lyubeznik, *F-modules: applications to local cohomology and D-modules in characteristic* $p > 0$, J. Reine Angew. Math. **491** (1997), 65–130. 220
[97] G. Lyubeznik, *On Bernstein-Sato polynomials*, Proc. Amer. Math. Soc. **125** (1997), 1941–1944. 236
[98] G. Lyubeznik, *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case*, Comm. Alg. **28** (2000), 5867–5882. 227
[99] G. Lyubeznik, *On the vanishing of local cohomology in characteristic* $p > 0$, Preprint (2004). 221
[100] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. 120
[101] I. Madsen and J. Tornehave, *From calculus to cohomology: de Rham cohomology and characteristic classes*, Cambridge University Press, Cambridge, 1997. 13
[102] T. Marley, *The associated primes of local cohomology modules over rings of small dimension*, Manuscripta Math. **104** (2001), 519–525. 228
[103] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. 250
[104] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1986. 5, 53, 170, 173, 250
[105] L. Matusevich, E. Miller, and U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc., to appear. `arXiv:math.AG/0406383` 239, 246, 248
[106] P. McMullen, *The maximum numbers of faces of a convex polytope*, Mathematika **17** (1970), 179–184. 152
[107] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics **227**, Springer, New York, 2004. 156, 157, 158, 196, 205, 207
[108] J. Milne, *Lectures on étale cohomology*, available at `http://www.jmilne.org/math`. 195
[109] J. Milnor, *Morse theory*, Annals of Mathematics Studies **51**, Princeton University Press, Princeton, New Jersey, 1963. 185

[110] T. T. Moh, *Set-theoretic complete intersections*, Proc. Amer. Math. Soc. **94** (1985), 217–220. 115

[111] S. Morita, *Geometry of differential forms*, Translations of Mathematical Monographs **201**, American Mathematical Society, Providence, Rhode Island, 2001. 13

[112] T. Oaku, *An algorithm of computing b-functions*, Duke Math. J. **87** (1997), 115–132. 233

[113] T. Oaku and N. Takayama, *An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation*, in: Effective methods in algebraic geometry (Saint-Malo, 1998), J. Pure Appl. Algebra **139** (1999), 201–233. 190

[114] T. Oaku and N. Takayama, *Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups*, J. Pure Appl. Algebra **156** (2001), 267–308. 238

[115] A. Ogus, *Local cohomological dimension of algebraic varieties*, Ann. of Math. (2) **98** (1973), 327–365. 143

[116] P. Orlik and H. Terao, *Arrangements and hypergeometric integrals*, MSJ Memoirs **9**, Mathematical Society of Japan, Tokyo, 2001. 183

[117] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 47–119. 61, 143, 211, 213, 217, 218

[118] D. Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985), 97–126. 31

[119] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171. 31

[120] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 103–106.

[121] P. Roberts, *Homological invariants of modules over commutative rings*, Séminaire de Mathématiques Supérieures **72**, Presses de l'Université de Montréal, Montreal, 1980.

[122] P. Roberts, *Rings of type* 1 *are Gorenstein*, Bull. London Math. Soc. **15** (1983), 48–50.

[123] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 177–180. 214

[124] P. C. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics **133**, Cambridge University Press, Cambridge, 1998. 191

[125] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen*, Invent. Math. **14** (1971), 123–142. 234

[126] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics **6**, Springer-Verlag, Berlin, 2000. 160, 162, 241

[127] M. Sato, T. Kawai, and M. Kashiwara, *Microfunctions and pseudo-differential equations*, in: Hyperfunctions and pseudo-differential equations (Katata, 1971), 265–529, Lecture Notes in Math. **287**, Springer, Berlin, 1973. 164

[128] J.-P. Serre, *Quelques problèmes globaux relatifs aux variétés de Stein*, in: Colloque sur les fonctions de plusieurs variables (Bruxelles, 1953), 57–68, Georges Thone, Liège; Masson & Cie, Paris, 1953. 185

[129] J.-P. Serre, *Une propriété topologique des domaines de Runge*, Proc. Amer. Math. Soc. **6** (1955), 133–134. 185

[130] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2), **61** (1955), 197–278. 120

[131] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–1956), 1–42. 184

[132] J.-P. Serre, *Sur la cohomologie des variétés algébriques*, J. Math. Pures Appl. (9) **36** (1957), 1–16. 128

[133] J.-P. Serre, *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000. 86, 87

[134] A. K. Singh, *p-torsion elements in local cohomology modules*, Math. Res. Lett. **7** (2000), 165–176. 228

[135] A. K. Singh, *p-torsion elements in local cohomology modules. II*, in: Local cohomology and its applications (Guanajuato, 1999), 155–167, Lecture Notes in Pure and Appl. Math. **226** Dekker, New York, 2002. 228

[136] A. K. Singh and I. Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. **33** (2004), 1703–1733. 230, 231

[137] A. K. Singh and U. Walther, *On the arithmetic rank of certain Segre products*, Contemp. Math., to appear. 112

[138] K. E. Smith, *Tight Closure of parameter ideals*, Invent. Math. **115** (1994), 41–60. 213

[139] R. Stanley, *The upper bound conjecture and Cohen–Macaulay rings*, Studies in Applied Math. **54** (1975), 135–142. 152
[140] R. P. Stanley, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics **41**, Birkhäuser, Boston, MA, 1996. 157, 197
[141] B. Sturmfels and N. Takayama, *Gröbner bases and hypergeometric functions*, in: Gröbner bases and applications (Linz, 1998), 246–258, London Math. Soc. Lecture Note Ser. **251**, Cambridge Univ. Press, Cambridge. 243
[142] A.A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), 1063–1066. 31
[143] R. G. Swan, *Néron-Popescu desingularization*, Algebra and geometry (Taipei, 1995), Lect. Algebra Geom. **2**, 135–192, Internat. Press, Cambridge, MA, 1998. 31
[144] U. Walther, *Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties*, in: Effective methods in algebraic geometry (Saint-Malo, 1998), J. Pure Appl. Algebra **139** (1999), 303–321. 220, 237, 238
[145] U. Walther, *Algorithmic computation of de Rham cohomology of complements of complex affine varieties*, in: Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998), J. Symbolic Comput. **29** (2000), 795–839. 191
[146] U. Walther, *Computing the cup product structure for complements of complex affine varieties*, in: Effective methods in algebraic geometry (Bath, 2000), J. Pure Appl. Algebra **164** (2001), 247–273. 191
[147] U. Walther, *Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements*, Compos. Math. **141** (2005), 121–145. 234
[148] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 1994. 13
[149] H. Weyl, *The concept of a Riemann surface*, Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964. 120
[150] H. Weyl, *The classical groups. Their invariants and representations*, Princeton University Press, Princeton, New Jersey, 1997. 89
[151] K. Yanagawa, *Sheaves on finite posets and modules over normal semigroup rings*, J. Pure Appl. Algebra **161** (2001), 341–366. 207
[152] T. Yano, *On the theory of b-functions*, Publ. Res. Inst. Math. Sci. **14** (1978), 111–202.
[153] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics **152**, Springer–Verlag, New York, 1995. 149, 151, 152