NON-COMMUTATIVE DESINGULARIZATIONS OF DETERMINANTAL VARIETIES

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ABSTRACT. I will define and give some background on what it should mean to be a "non-commutative resolution of singularities," with emphasis on examples. Then I'll explain joint work with Buchweitz and Van den Bergh showing existence of such desingularizations for a large class of rings defined by minors of generic matrices.

Let's begin with the definition.

Definition (Van den Bergh '04)**.** Let R be a Gorenstein normal domain. A *non-commutative crepant resolution (of singularities)* of R (or $\operatorname{Spec} R$) is an R-algebra of the form

$$\mathcal{E} = \operatorname{End}_R(M)$$

such that

- M is a finitely generated reflexive R-module;
- $\operatorname{gldim} \mathcal{E} < \infty$;
- $\mathcal{E}$ is a maximal Cohen–Macaulay R-module.

One of my main goals in these talks is to convince you that this is a reasonable and natural definition, so I won't give much in the way of motivation right now. Let's make sure, though, that we understand all the pieces of the definition.

- *Gorenstein* means $\operatorname{injdim} R < \infty$. Equivalently, R is its own canonical module. Easiest case to think about: suppose R is complete local and contains a field k. Then by Cohen's Structure Theorem, R is a finitely generated module over some power series ring $T = k[[x_1, \ldots, x_n]] \subseteq R$. To say that R is Gorenstein means that $\operatorname{Hom}_T(R, T) \cong R$.

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sketchy lecture notes. use at your own risk.

- *Normal* means that R is integrally closed. Equivalently (since R is CM by above) R is regular in codimension one, that is, $\operatorname{Spec} R$ has only singularities of codimension $\geqslant 2$. This is a relatively minor condition, since (as long as R is "excellent") the integral closure of R is a finite R-module, so we can replace R by its integral closure with few consequences. This is usually the first step in any "resolution of singularities." (Notice that if R has dimension $\leqslant 1$, then R normal implies that R is regular, which is the uninteresting case for us. If R has dimension 2, normality means it's an isolated singularity.)
- An R-module M is *reflexive* if the natural biduality map $M \longrightarrow M^{**}$, sending $m \in M$ to "evaluation at m," is bijective. This is also a relatively minor condition: we need to assume at least that M is torsionfree (so the map is injective), to rule out cases like $\operatorname{End}_R(k) = k$, and then it's a short step to reflexivity.
- *Global dimension* of $\mathcal{E}$ is the supremum of injective (or projective) dimensions of finitely generated $\mathcal{E}$-modules. For commutative rings, global dimension is finite if and only if the ring is regular, but it's a much weaker condition for non-commutative rings. That's why we have the third condition.
- N is a *maximal Cohen-Macaulay* R-module if $\operatorname{depth} N = \dim R$, the largest allowed value. (Depth is the maximal length of a regular sequence.) Since R is normal and of dimension $\geqslant 2$, MCM modules are automatically reflexive; the converse is only true in dimension 2.

Let's go ahead and see how something satisfying this definition arises in nature.

Example (Quotient Singularities)**.** Let $S = k[[x_1, \ldots, x_n]]$, with k a field, and let $G \subseteq \mathrm{GL}_n(k)$ be a finite group, acting on S by linear changes of variables. We insist that $|G|$ be invertible in k (otherwise everything breaks horribly).

Let $R = S^G$, the ring of invariants. Then R is a local normal domain of dimension n, and is Cohen–Macaulay by the Hochster–Roberts Theorem.

Assume that G contains no *pseudo-reflections*, that is, elements having 1 as an eigenvalue of multiplicity $n-1$. (E.g., $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ acting on $k[[x,y]]$.) This is no big restriction, since...

Theorem (Shephard–Todd '54, Chevalley '55)**.** *The group G is generated by pseudo-reflections if and only if R is a regular ring.*

Also, once we rule out pseudo-reflections, we can use:

Theorem (Watanabe '74)**.** *Suppose G contains no pseudo-reflections. Then R^G is Gorenstein if and only if $G \subseteq \mathrm{SL}_n(k)$.*

So we now additionally assume $G \subseteq \mathrm{SL}$, so that R is a *Gorenstein* normal domain.

Here's a particular example to keep in mind: Let $S = \mathbb{C}[[x,y]]$ and $G = \mathbb{Z}/2\mathbb{Z}$, with the generator acting by $x \mapsto -x$ and $y \mapsto -y$. Then $R = S^G = \mathbb{C}[[x^2, xy, y^2]] \cong \mathbb{C}[[a,b,c]]/(b^2 - ac)$. In this case, we have $S \cong R \oplus (x,y)$ as R-modules (the $0 \bmod 2$ stuff and the $1 \bmod 2$ stuff).

In general, S is always a maximal Cohen-Macaulay R-module. In particular, it's a reflexive R-module.

Theorem (Auslander '62)**.** *In this setup, the endomorphism ring $\mathcal{E} = \mathrm{End}_R(S)$ is isomorphic to the* twisted group ring

$$S\#G = \bigoplus_{g \in G} Sg, \qquad (sg)(th) = st^g gh\,.$$

In particular, $\mathcal{E} \cong S^{|G|}$ as R-modules, so is MCM. Also,

$$\mathrm{Ext}^i_{\mathcal{E}}(-,-) \cong \mathrm{Ext}^i_S(-,-)^G$$

so that $\mathcal{E}$ has finite global dimension, equal to that of S. (The display is true more or less by definition for $i = 0$, and then true for higher Exts since $-^G$ is an exact functor.) Thus $\mathcal{E}$ is a non-commutative crepant resolution of R.

Much more is true in the special case $n = 2$. Here, S is isomorphic to the direct sum of *all* indecomposable MCM (reflexive) R-modules (so in particular there are only finitely many) by a theorem of J. Herzog, and we get one-one correspondences (the *McKay Correspondence*) between those modules, irreducible representations of G, and components in the exceptional fibre of the canonical resolution of singularities. That's a topic for another series of talks.

Example (2×2 determinant)**.** Let $S = \mathbb{C}[[x, y, z, w]]$ and $R = S/(xw - zy)$. Then R is a three-dimensional Gorenstein normal domain.

If we think of $\operatorname{Spec} S = \mathbb{A}^4_{\mathbb{C}}$ as the space of matrices $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ with entries from $\mathbb{C}$, then $X = \operatorname{Spec} R$ is the subset of such matrices with vanishing determinant.

There are two "obvious" resolutions of singularities of X: we can blow up either of the height-one primes

$$\mathfrak{p} = (x, y) \qquad \text{or} \qquad \mathfrak{q} = (x, z)\,.$$

(Recall that, for example, the blowup at $\mathfrak{p}$ is obtained as follows. Set $Q = R[\mathfrak{p}t] = R[xt, yt]$. Then

$$Q \cong \mathbb{C}[[x, y, z, w, a, b]]/(xw - zy, xb - ya)\,,$$

and the blowup is $\operatorname{Bl}_{\mathfrak{p}}(X) = \operatorname{Proj} Q$, the set of prime ideals of Q that don't contain (a, b). This space is *smooth*: if a prime $\mathfrak{P}$ doesn't contain a, then in the localization at $\mathfrak{P}$, a is a unit, so we can solve for y in terms of x and see the localization as a union of two smooth components. Similarly for b.)

Notice that both $\mathfrak{p}$ and $\mathfrak{q}$ are MCM R-modules (since, e.g., the quotient $R/\mathfrak{p} \cong \mathbb{C}[[z, w]]$ has depth 2, forcing $\mathfrak{p}$ to have depth 3).

Also, $\mathfrak{q} = \mathfrak{p}^{-1}$ as fractional ideals (R-submodules of the quotient field). To see this, notice that

$$\begin{aligned} \mathfrak{p}\mathfrak{q} &= (x, y)(x, z) \\ &= (x^2, xy, xz, yz) \\ &= x(x, y, z, w) \\ &\cong (x, y, z, w) \end{aligned}$$

so that $(\mathfrak{p}\mathfrak{q})^{**} = R$. Alternatively, $\mathfrak{q} \cong \mathfrak{p}^* = \mathrm{Hom}_R(\mathfrak{p}, R)$.

Remark: These are two special cases of the *Springer resolutions*, which we'll see more of later.

Set

$$\mathcal{E} = \mathrm{End}_R(R \oplus \mathfrak{p})$$

Then

$$\begin{aligned}\mathcal{E} &\cong \begin{bmatrix} R & \mathfrak{p} \\ \mathfrak{p}^* & \mathrm{Hom}_R(\mathfrak{p}, \mathfrak{p}) \end{bmatrix} \\ &\cong \begin{bmatrix} R & \mathfrak{p} \\ \mathfrak{q} & R \end{bmatrix}\end{aligned}$$

as R-modules. (In this notation we understand the multiplication to use the isomorphism $\mathfrak{q} = \mathfrak{p}^*$.) So $\mathcal{E}$ is obviously MCM as an R-module.

Further, $\mathcal{E}$ has global dimension 3. (This is a short calculation, explicitly resolving the simple modules, which I'll skip.) Therefore $\mathcal{E}$ is a non-commutative crepant resolution of R.

So we've at least seen that this definition of "desingularization" arises in a couple of places in nature. But why should it be called a desingularization?

A *resolution of singularities* (desingularization) of a scheme X is a map $\pi\colon \widetilde{X} \longrightarrow X$ so that $\widetilde{X}$ is smooth, π is *proper* and *birational*.

Such a thing is *crepant* if

$$\pi^* K_X = K_{\widetilde{X}}\,.$$

where K is the canonical divisor, so $\omega_X = \bigwedge^{\dim X} \Omega^1$ is the bundle of highest differential forms, and $\omega_X = \mathcal{O}_X(K_X)$. (Morally speaking, tensoring the bundle of highest differential forms on X up to $\widetilde{X}$ gives exactly the highest differential forms up there. In general we have $K_{\widetilde{X}} = \pi^* K_X + \Delta$ for some *discrepancy divisor* Δ.)

Note that since R (i.e. $X = \operatorname{Spec} R$) is Gorenstein, ω_X is locally free, so an honest line bundle.

(Aside: In terms of Karl Schwede's talk, schemes admitting crepant resolutions are *canonical* but in general not *terminal*. In particular they have rational singularities.)

Now: we interpret the definition of $\mathcal{E}$ as follows.

$\mathcal{E} = \mathrm{End}_R(M)$	$\mathcal{E}$ is a finitely generated R-module	"proper"
$\mathcal{E} = \mathrm{End}_R(M)$	$\mathcal{E} \otimes_R K \cong M_n(K)$, where K is the quotient field	"birational"
$\mathrm{gldim}\, \mathcal{E} < \infty$	$\mathcal{E}$ is nonsingular	"smooth"
$\mathcal{E}$ MCM / R	$\mathrm{Ext}_R^{>0}(\mathcal{E}, R) = 0$, so $\mathcal{E}$ is its own "dualizing module"	"crepant"

Actually, nothing in this table indicates why we should expect $\mathcal{E}$ to be an endomorphism ring, rather than just a module-finite R-algebra of finite global dimension, satisfying the birationality and MCM conditions. But suppose $\mathcal{A}$ is such an algebra. Then one can show that for each height-one prime $\mathfrak{p}$ of R, $\mathcal{A}_\mathfrak{p}$ is a maximal order in $M_n(K)$ over the DVR $R_\mathfrak{p}$ (in the sense of Auslander-Goldman). It follows that

$$\omega_\mathcal{A} = \omega_R \otimes \left(\bigotimes_\mathfrak{p} \mathfrak{p}^{-1+\text{ramification index of } \mathcal{A}_\mathfrak{p}} \right)^{**} .$$

So if we want $\omega_\mathcal{A}$ to be generated by ω_R (analogous to the crepancy condition) then we need all the ramification indices to be 1. That says $\mathcal{A}$ is a maximal order in $M_n(K)$. By Auslander-Goldman, this implies $\mathcal{A} \cong \mathrm{End}_R(M)$ for some reflexive M.

So here's the actual new content.

Theorem. *Let K be a field, $X = (x_{ij})$ an $m \times n$ matrix of indeterminates over K with $n \geqslant m$, $S = K[X]$ the polynomial ring, $I_m(X)$ the ideal of maximal minors of X, and $R = S/I_m(X)$. Then R has a non-commutative crepant resolution.*

Remark.

(1) R is a normal domain of dimension $mn - n + m - 1$.

(2) R is Gorenstein if and only if $m = n$, that is, the matrix is square, and so $I_m(X) = \det(X)$. Properly speaking, this means that we are not allowed to use the phrase "non-commutative crepant resolution" in the general situation. We will anyway, and understand that for the purposes of this talk, we accept the definition verbatim for all Cohen–Macaulay normal domains. (This is a bad idea, for reasons we won't get into.)

(3) The case $m = n = 2$ was our second example. It's the only case where R is an isolated singularity.

(4) The matrix X determines the *generic S-linear map* $\varphi\colon S^n \longrightarrow S^m$. To set notation for later, let's set $\mathcal{G} = S^n$ and $\mathcal{F} = S^m$.

(5) Thinking of $\operatorname{Spec} S$ as $\mathbb{A}_K^{mn} \cong \operatorname{Mat}_{m\times n}(K)$, $\operatorname{Spec} R$ consists of the locus where φ drops rank.

To prove the Theorem, I need to construct an R-module M so that $E = \operatorname{End}_R(M)$ has the desired properties. Here's how.

Construction. Define an S-module M_1 by the exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \longrightarrow M_1 \longrightarrow 0\,.$$

Take exterior powers of this sequence to define M_a

$$0 \longrightarrow \bigwedge^a \mathcal{G} \longrightarrow \bigwedge^a \mathcal{F} \longrightarrow M_a \longrightarrow 0\,.$$

for $a = 1, \ldots, m$. Notice that $\bigwedge^m \mathcal{F} \cong S$, and $\bigwedge^m \varphi$ is given by the matrix of m-minors of X, so that $M_m \cong R$.

Fact. Each M_a is killed by $I_m(X)$, so is actually an R-module. Moreover, the minimal (graded) free resolution of M_a is given by the Buchsbaum-Rim complex, so one can compute the depth and see that each M_a is a MCM R-module.

Set $M = M_1 \oplus \cdots \oplus M_m$ and $E = \operatorname{End}_R(M)$.

Theorem. *E is MCM as an R-module. Equivalently, $\operatorname{Hom}_R(M_b, M_a)$ is MCM for all a, b. Furthermore, E has finite global dimension.*

Parts of the Proof. The idea is to translate S, R, M_a, etc., into geometry.

Definition.

- Let G and F be K-vector spaces of ranks n and m, resp. Think of $\mathcal{G}$ and $\mathcal{F}$ as $G \otimes_K S$ and $F \otimes_K S$, resp.
- Identify $\operatorname{Spec} S$ with the vector space $\operatorname{Hom}_K(G, F)$ and $\operatorname{Spec} R$ with the linear maps of rank $< m$.
- Resolve the singularities of $\operatorname{Spec} R$ as follows.
  - Let $\mathbb{P} = \mathbb{P}(F^*) \cong \mathbb{P}^{m-1}$ be the projective space on F^*, the dual of F.
  - Let $\mathcal{Y} = \operatorname{Hom}_K(G, F) \times \mathbb{P}$.
  - Define

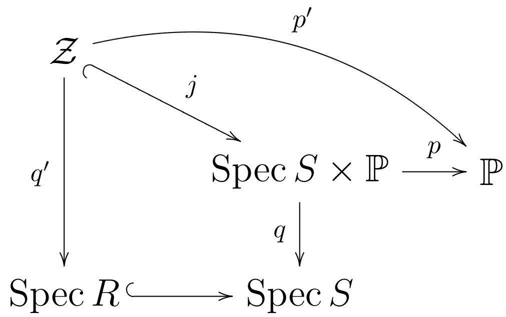
$$\mathcal{Z} = \{(\theta, [\lambda]) \in \operatorname{Hom}_K(G, F) \times \mathbb{P}(F^*) \mid \lambda\theta = 0\} \ .$$

Remark. Notice that $\lambda\theta\colon G \longrightarrow F \longrightarrow K$ is only defined up to a scalar multiple, but its zero-ness is well-defined.

We have a natural projection $\mathcal{Z} \longrightarrow \operatorname{Hom}_K(G, F)$, sending $(\theta, [\lambda])$ to θ. Claim that the image is precisely $\operatorname{Spec} R$. To see this, note $\theta \in \operatorname{Spec} R$ if and only if $\operatorname{rank} \theta < m$, if and only if $\operatorname{image} \theta$ is contained in some hyperplane in F, if and only if there is a nonzero $\lambda\colon F \longrightarrow K$ that kills $\operatorname{image} \theta$.

Finally, it's not hard to check that $\mathcal{Z}$ is a smooth scheme. It's locally defined by n quadrics in $\mathcal{Y}$, and is in fact a complete intersection in $\mathcal{Y}$.

So here's the picture.



Plan. Identify sheaves $\mathcal{M}_a$ on $\mathbb{P}(F^*)$ so that $q_* p^* \mathcal{M}_a = M_a$, and control the cohomology so that we can conclude $q_* p^* \mathscr{H}om_{\mathcal{O}_\mathcal{Z}}(\mathcal{M}_b, \mathcal{M}_a) = \operatorname{Hom}_R(M_b, M_a)$.

Definition. The *sheaf of Kähler differentials* $\Omega_{\mathbb{P}(F^*)}$ on $\mathbb{P}(F^*)$ is defined by the exact sequence

$$0 \longrightarrow \Omega_\mathbb{P} \longrightarrow F \otimes_K \mathcal{O}_\mathbb{P}(-1) \longrightarrow \mathcal{O}_\mathbb{P} \longrightarrow 0\,.$$

In other words, it's the first syzygy in the Koszul complex on $\mathbb{P}$. The *higher* differentials $\Omega^a_\mathbb{P}$, for $a = 0, \ldots, m-1$, are defined by taking exterior powers of this sequence, which gives

$$0 \longrightarrow \Omega^a \longrightarrow \bigwedge^a F \otimes_K \mathcal{O}_\mathbb{P}(-a) \longrightarrow \Omega^{a-1} \longrightarrow 0\,.$$

In particular, we have $\Omega^0 = \mathcal{O}_\mathbb{P}$, $\Omega^1 = \Omega$, $\Omega^{m-1} = \mathcal{O}_\mathbb{P}(-m)$, and $\Omega^m = 0$.

Theorem. *Define*

$$\mathcal{T} = p^* \left(\Omega^0(1) \oplus \Omega^1(2) \oplus \cdots \oplus \Omega^{m-1}(m)\right)\,,$$

a locally free sheaf on $\mathcal{Z}$. Then $\mathcal{T}$ is a tilting object *in the derived category $\mathcal{D}^b(\mathcal{Z})$, that is,*

(1) $\operatorname{Hom}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T}, \mathcal{T}[i]) = 0$ *for* $i \neq 0$.

(2) $\mathcal{T}$ generates $\mathcal{D}^b(\mathcal{Z})$.

In particular, this implies that

$$\operatorname{Hom}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T}, -)\colon \mathcal{D}^b(\mathcal{Z}) \longrightarrow \mathcal{D}^b(\operatorname{End}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T}))$$

is an exact equivalence of categories. In particular, $\operatorname{End}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T})$ *has finite global dimension since $\mathcal{Z}$ does.*

Given this, all we need is the following theorem.

Theorem.

(1) $q_*p^*\Omega^{a-1}(a) \cong M_a$ *as* S*-modules for* $a = 1, \ldots, m$*. Furthermore,* $\mathbf{R}^i q_* \mathcal{T} = 0$ *for* $i > 0$*. In combination with the known resolution of* $\Omega^{a-1}(a)$ *over* $\mathbb{P}$ *(just the Koszul complex!), this implies that* $\operatorname{Hom}_R(M_b, M_a)$ *is MCM for all* a, b*.*

(2) Pushforward via q *induces* $\operatorname{Hom}_{\mathcal{O}_{\mathcal{Z}}}(\Omega^{b-1}(b), \Omega^{a-1}(a)) \cong \operatorname{Hom}_R(M_b, M_a)$*. In particular,* $\operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}) \cong \operatorname{End}_R(M)$*, and so the latter has finite global dimension.*

Part 3.

From where I left off last time, there are two obvious directions to go. One gets very technical very quickly: proving that a certain vector bundle on the desingularization $\mathcal{Z}$ is tilting, then computing lots of cohomology to prove a certain algebra gives a non-commutative crepant resolution. Here's the other.

We begin with a baby case, referred to as *Beilinson's tilting description of* $D^b(\operatorname{coh}\mathbb{P})$.

Let K be a field and F a K-vector space of rank m. Set $\mathbb{P} = \mathbb{P}(F^*) \cong \mathbb{P}^{m-1}_K$. Let $\Omega_{\mathbb{P}}$ be the sheaf of Kähler differentials on $\mathbb{P}$, introduced last time. So $\Omega_{\mathbb{P}}$ is defined by the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}} \longrightarrow F \otimes_K \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0\,, \tag{1}$$

which is the beginning of the sheafified Koszul complex of the homogeneous coordinate ring of $\mathbb{P}$.

Let $\Omega^a = \bigwedge^a \Omega_{\mathbb{P}}$, the sheaf of a-forms on $\mathbb{P}$. They're defined by taking exterior powers of 1:

$$0 \longrightarrow \Omega^a \longrightarrow \bigwedge^a F \otimes_K \mathcal{O}_{\mathbb{P}} \longrightarrow \Omega^{a-1} \longrightarrow 0\,. \tag{2}$$

In fact, these sequences splice together (after twisting appropriately) to give the full Koszul complex on $\mathbb{P}$:

$$0 \longrightarrow \bigwedge^m F \otimes_K \mathcal{O}_{\mathbb{P}}(-m) \longrightarrow \cdots \longrightarrow F \otimes_K \mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0\,. \tag{3}$$

In particular $\Omega^{m-1} \cong \bigwedge^m F \otimes_K \mathcal{O}_{\mathbb{P}}(-m) \cong \mathcal{O}_{\mathbb{P}}(-m)$, and $\Omega^0 \cong \mathcal{O}_{\mathbb{P}}$.

Lemma (Beilinson '78)**.** For all a, b with $1 \leqslant a, b \leqslant m$, and all $i \geqslant 1$, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{P}}(\Omega^{b-1}(b), \Omega^{a-1}(a)) &&= \textstyle\bigwedge^{b-a} F\\
&\mathrm{Ext}^i_{\mathbb{P}}(\Omega^{b-1}(b), \Omega^{a-1}(a)) &&= 0\\
&\mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-b+1), \mathcal{O}_{\mathbb{P}}(-a+1)) &&= \mathrm{Sym}_{b-a} F\\
&\mathrm{Ext}^i_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-b+1), \mathcal{O}_{\mathbb{P}}(-a+1)) &&= 0
\end{aligned}$$

(We don't care much about the third and fourth lines, but they give analogues of the results below for the line bundles $\mathcal{O}_{\mathbb{P}}, \ldots, \mathcal{O}_{\mathbb{P}}(-m)$. Also, I've introduced some ± 1's here for my own purposes; they don't affect anything on the right-hand side, so can mostly be ignored, if you're willing to be flexible about the indexing.)

Beilinson then proves that $\Omega^0(1), \ldots, \Omega^{m-1}(m)$ generated the bounded derived category $D^b(\mathrm{coh}\,\mathbb{P})$, and concludes [by resolving the diagonal $\Delta \subseteq \mathbb{P} \times \mathbb{P}$]

Theorem. *The functor*

$$\mathbf{R}\mathrm{Hom}_{\mathbb{P}}\left(\bigoplus_{a=1}^{m} \Omega^{a-1}(a), --\right) : D^b(\mathrm{coh}\,\mathbb{P}) \longrightarrow D^b(\mathrm{mod}\,\mathrm{End}_{\mathbb{P}}\left(\bigoplus_{a=1}^{m} \Omega^{a-1}(a)\right))$$

is an equivalence of derived categories.

Note. Let's call the ring on the right B. By the Lemma, B can be interpreted as a matrix ring:

$$\mathrm{End}_{\mathbb{P}}\left(\bigoplus_{a=1}^{m} \Omega^{a-1}(a)\right) = \begin{bmatrix} K & F & \cdots & \cdots & \bigwedge^{m-1} F \\ 0 & K & F & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & K & F \\ 0 & 0 & \cdots & \cdots & K \end{bmatrix}$$

There's a similar version for $\bigoplus \mathcal{O}_{\mathbb{P}}(-a)$ involving symmetric powers.

Thus B is a *quiverized* version of the exterior algebra $\bigwedge^\bullet F$. It has m idempotents, corresponding to the endomorphism that's the identity on the a^{th} summand and zero elsewhere. To draw a quiver, use the idempotents as vertices, and for each basis vector λ_i of F, draw $m-2$ arrows labeled λ_i, one each from a to $a-1$:

$$1 \;\begin{array}{c} \xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow[\quad\lambda_1\quad]{} \end{array}\; 2 \;\begin{array}{c} \xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow[\quad\lambda_1\quad]{} \end{array}\; \cdots\cdots \;\begin{array}{c} \xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow[\quad\lambda_1\quad]{} \end{array}\; m$$

Now impose the relations

$$\lambda_i\lambda_j + \lambda_j\lambda_i = 0 = \lambda_i^2$$

for all $1 \leqslant a, b \leqslant m$, to reflect the fact that $\operatorname{Hom}_{\mathbb{P}}(\Omega^{a-1}(a), \Omega^{a+1}(a+2)) \cong \bigwedge^2 F$.

We call this the *Beilinson quiver of rank* m. It follows from Beilinson's Lemma that its path algebra with relations is isomorphic to B.

Now let G be another K-vector space of rank $n \geqslant m$, and return to the notation from last time: $\operatorname{Spec} S = \operatorname{Hom}_K(G, F)$, $\mathcal{G} = S \otimes_K G$, $\mathcal{F} = S \otimes_K F$, and $\varphi\colon \mathcal{G} \longrightarrow \mathcal{F}$ is the generic S-linear map, with $R = S/I_m(\varphi)$. Recall that we defined R-modules $M_1, \ldots, M_m$ by

$$\bigwedge^a \mathcal{G} \xrightarrow{\bigwedge^a \varphi} \bigwedge^a \mathcal{F} \longrightarrow M_a \longrightarrow 0$$

Then we decided last time that we care about $E = \operatorname{End}_R(M)$, where $M = \bigoplus_{a=1}^m M_a$.

An endomorphism of M is given by a commutative diagram

$$\begin{array}{ccccccc} \bigwedge^\bullet \mathcal{G} & \xrightarrow{\bigwedge^\bullet \varphi} & \bigwedge^\bullet \mathcal{F} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \vdots & & \\ \bigwedge^\bullet \mathcal{G} & \xrightarrow{\bigwedge^\bullet \varphi} & \bigwedge^\bullet \mathcal{F} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Here are three examples of such endomorphisms:

(1) For $\lambda \in \mathcal{F}^*$, we get a Koszul-type contraction map $\partial_\lambda\colon \bigwedge^a \mathcal{F} \longrightarrow \bigwedge^{a-1} \mathcal{F}$. Similarly, we get $\partial_{\varphi\circ\lambda}\colon \bigwedge^a \mathcal{G} \longrightarrow \bigwedge^{a-1} \mathcal{G}$. These give "degree -1" endomorphisms $\partial_\lambda\colon M \longrightarrow M$.

(2) For $g \in \mathcal{G}$, we get multiplication maps $\mu_g\colon \bigwedge^a \mathcal{G} \longrightarrow \bigwedge^{a+1} \mathcal{G}$ and $\mu_{\varphi(g)}\colon \bigwedge^a \mathcal{F} \longrightarrow \bigwedge^{a+1} \mathcal{F}$, which give "degree $+1$" maps $\mu_g\colon M \longrightarrow M$.

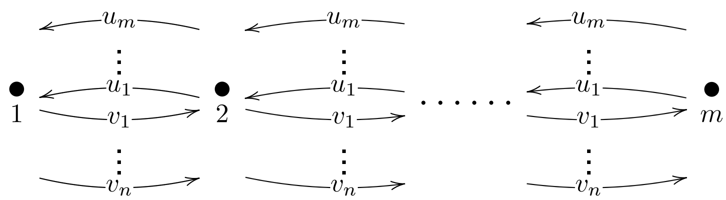
(3) Finally, we have the projectors $e_a\colon M \twoheadrightarrow M_a \hookrightarrow M$ for $a = 1, \ldots, m$, which come from similar projectors on $\bigwedge^\bullet \mathcal{F}$ and $\bigwedge^\bullet \mathcal{G}$.

Fixing bases $\lambda_1, \ldots, \lambda_m$ for $\mathcal{F}$ and $g_1, \ldots, g_n$ for $\mathcal{G}$, we write $\partial_1, \ldots, \partial_m$ and $\mu_1, \ldots, \partial_n$ for the induced maps.

There are some relations among these endomorphisms:

$$\partial_i\partial_j + \partial_j\partial_i = 0 = \partial_i^2$$
$$\mu_i\mu_j + \mu_j\mu_i = 0 = \mu_i^2$$
$$\partial_i\mu_j + \mu_j\partial_i = \partial_i(\varphi(g_j)) = x_{ij}$$

Definition. Define the *quiverized Clifford Algebra* to be the path algebra over S of the following quiver with m vertices:



subject to the relations

- $u_iu_j + u_ju_i = u_i^2 = v_iv_j + v_jv_i = v_i^2 = 0$
- $u_iv_j + v_ju_i = x_{ij}$ (the "Clifford relation").

Precisely, $\mathcal{C}$ is the quotient of the polynomial ring $S\langle e_1, \ldots, e_n; u_1, \ldots, u_n; v_1, \ldots, v_n\rangle$ by the two-sided ideal generated by the relations

- $e_ae_b = \delta_{ab}e_a$, $\sum e_a = 1$
- $v_ie_a = e_{a+1}v_i$
- $u_ie_a = e_{a-1}u_i$
- $u_iu_j + u_ju_i = u_i^2 = v_iv_j + v_jv_i = v_i^2 = 0$
- $u_iv_j + v_ju_i = x_{ij}$ (the "Clifford relation").

We call this quiver the *doubled Beilinson quiver,* for obvious reasons. Its path algebra $\mathcal{C}$ is, in a sense, two quiverized exterior algebras woven together by the Clifford relation.

By the way we obtained those relations, we have an action of $\mathcal{C}$ on M, so a homomorphism $\mathcal{C} \longrightarrow \mathrm{End}_R(M)$. By translating to geometry again, and showing that there is a map $\mathcal{C} \longrightarrow \mathrm{End}_{\mathcal{Z}}(\mathcal{T})$ inducing this one, we show

Theorem. *The induced map $\mathcal{C} \longrightarrow \mathrm{End}_R(M)$ is an isomorphism.*

It's relatively easy to prove that $\mathcal{C}$ is a maximal Cohen–Macaulay R-module: we write down a resolution for $\mathcal{C}$ over the path algebra of an infinite version of the quiver.

Don't do this unless you're way short of time.

One last result, tying the geometry together with the quiver. Since the desingularization $\mathcal{Z}$ and the non-commutative desingularization E are derived equivalent, $\mathcal{Z}$ parametrizes certain objects in the derived category of E, which is the derived category of $\mathcal{C}$. So there are certain quiver representations corresponding to the points of $\mathcal{Z}$.

Theorem. *The variety $\mathcal{Z}$ is the fine moduli space for the representations V of the doubled Beilinson quiver of the following sort:*

- *V has dimension vector $(1, m-1, \binom{m-1}{2}, \ldots, 1)$, and*
- *V is generated as a $\mathcal{C}$-module by V_m.*

The simple representations are precisely those parametrized by points lying over the nonsingular locus of $\operatorname{Spec} R$.