FINITE COHEN–MACAULAY TYPE

by

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Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of Krull dimension d. A non-zero R-module M is *maximal Cohen–Macaulay* (MCM) provided it is finitely generated and there exists an M-regular sequence $x_1, \ldots, x_d$ in the maximal ideal $\mathfrak{m}$. In particular, R is a *Cohen–Macaulay* (CM) ring if R is a MCM module over itself. The ring R is said to have *finite Cohen–Macaulay type* (or finite CM type) if there are, up to isomorphism, only finitely many indecomposable MCM R-modules.

The first part of this dissertation investigates the non-complete CM local rings of finite CM type. In this direction, the main focus has been on a conjecture of Schreyer [33], which states that a local ring R has finite CM type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite CM type. In Chapter 1, which contains joint work with R. Wiegand, I prove that finite CM type ascends to the completion for excellent CM local rings. In Chapter 2, I prove ascent of finite CM type when R is a CM local ring with a Gorenstein module, and such that the Henselization R^{h} is excellent.

The second part of this dissertation considers one-dimensional complete hypersurfaces of mixed characteristic. Such rings are of the form $R = V[[y]]/(f)$, where (V, π) is a complete discrete valuation ring of characteristic zero with algebraically closed residue field of prime characteristic p. Using the theory of Auslander–Reiten quivers, I prove that the obvious generalizations of the one-dimensional complete equicharacteristic hypersurfaces with finite CM type do indeed have finite CM type.

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Chapter 0

Introduction and Preliminaries

Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of Krull dimension d. A non-zero R-module M is *maximal Cohen–Macaulay* (MCM) provided it is finitely generated and there exists an M-regular sequence $x_1, \ldots, x_d$ in the maximal ideal $\mathfrak{m}$. In particular, R is a *Cohen–Macaulay* (CM) ring if R is a MCM module over itself. The ring R is said to have *finite Cohen–Macaulay type* (or finite CM type) if there are, up to isomorphism, only finitely many indecomposable MCM R-modules.

There has been a great deal of progress in recent years on the problem of classifying all local rings of finite CM type. Most of the work on this question has focused on the complete case. The capstone of this research is a beautiful theorem of Auslander [1]: A complete CM local ring of finite CM type is an isolated singularity. Another highlight of the theory in the complete case is due to Herzog [19]: A complete Gorenstein local ring of finite CM type is a hypersurface.

The complete equicharacteristic hypersurface singularities of finite CM type have been completely characterized ([3], [15], [16], [22], [37]). A complete equicharacteristic hypersurface singularity is a ring of the form $R = A/(f)$, where $A = k[[x_0, \ldots, x_d]]$ is the ring of formal power series over an algebraically closed field k and f is a nonzero element in the square of the maximal ideal of A. For $d \geq 1$ and $\mathrm{char}(k) \neq 2$ it is known that such a singularity has finite CM type if and only if $R \cong k[[x_0, \ldots, x_d]]/(g + x_2^2 + \cdots + x_d^2)$, where $g \in k[x_0, x_1]$ defines a simple plane curve singularity. For $\mathrm{char}(k) \neq 2, 3, 5$, these simple plane curve singularities are defined by the following polynomials, corresponding to certain Dynkin diagrams:

$$
\begin{array}{lll}
(\mathrm{A}_n) & x_0^2 + x_1^{n+1}, & (n \geq 1) \\
(\mathrm{D}_n) & x_1(x_0^2 + x_1^{n-2}), & (n \geq 4) \\
(\mathrm{E}_6) & x_0^3 + x_1^4 & \\
(\mathrm{E}_7) & x_0(x_0^2 + x_1^3) & \\
(\mathrm{E}_8) & x_0^3 + x_1^5. &
\end{array}
$$

The first part of this dissertation investigates the non-complete CM local rings of finite CM type. In this direction, the main focus has been on a conjecture of Schreyer [33], which states that a local ring R has finite CM type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite CM type. In [39], R. Wiegand proved that finite CM type satisfies faithfully flat descent, provided the closed fiber is Cohen–Macaulay, thereby establishing one direction of the conjecture. In Chapter 1, which contains joint work with R. Wiegand, I prove ascent of finite CM type for excellent CM local rings. That is, if R is an excellent CM local ring of finite CM type, then the completion $\widehat{R}$ also has finite CM type. The proof uses previous results of Wiegand to reduce to showing that R is Gorenstein on the punctured spectrum (that is, $R_\mathfrak{p}$ is Gorenstein for all nonmaximal primes $\mathfrak{p}$). A CM local ring R with canonical module ω is Gorenstein on the punctured spectrum if and only if ω is a d^{th} syzygy, where d is the Krull dimension of R. Since not every excellent CM local ring has a canonical module, we show that finite CM type (or something very similar, called finite syzygy type) passes to the Henselization R^{h}, which does have a canonical module by Artin approximation. Then basic properties of the canonical module (see Section 1.1) are used to show that R^{h} is Gorenstein on the punctured spectrum. The material in this chapter appears in [25].

In Chapter 2, I prove ascent of finite CM type when R is a CM local ring with a Gorenstein module, and such that the Henselization R^{h} is excellent. As before, the problem comes down to showing that a CM local ring with a Gorenstein module and with finite CM type is Gorenstein on the punctured spectrum. I extend results of S. Ding [7] to show that a CM local ring R is Gorenstein on the punctured spectrum if and only if R has finite index, in the sense of Auslander (see Section 2.2). It is straightforward (Section 2.3) to show that if R has finite CM type, then R has finite index. This shows that finite CM type ascends to the Henselization R^{h}, and thence to $\widehat{R}$ by a result of Wiegand (see Theorem 0.3).

It is worth pointing out the similarities and differences between the main results of Chapters 1 and 2. In both cases, I assume that the Henselization R^{h} is excellent, but for slightly different reasons. In Chapter 1, the excellence of R^{h} implies that R^{h} has a canonical module (see the discussion at the beginning of Section 1.2), while in Chapter 2 I do not use this information. In both chapters, the excellence of R^{h} is used to conclude that if R^{h} is an isolated singularity, then the completion $\widehat{R}$ is also an isolated singularity. Thus the main difference between the main results of Chapters 1 and 2 is that in Chapter 1 I prove that finite CM type ascends from R to R^{h} if R^{h} has a canonical module, while in Chapter 2 the assumption is that R have a Gorenstein module. It is known ([11, Cor. 4.8]) that a Henselian local ring with a Gorenstein module necessarily has a canonical module. Thus the results of Chapter 2 do not give completely new applications to the ascent of finite CM type. The methods, however, are completely different and perhaps of independent interest.

Chapter 2 closes with a few examples. The first two show that the main results of Chapters 1 and 2 can fail for non-Cohen–Macaulay local rings. Both of these are local rings of finite CM type such that the completions have infinite CM type. The third example, due to J.-I. Nishimura, is an excellent CM local ring with no Gorenstein module.

Chapter 3 furthers the program mentioned above of classifying complete hypersurfaces of finite CM type, by considering one-dimensional complete hypersurfaces of mixed characteristic. Such rings are of the form $R = V[[y]]/(f)$, where (V, π) is a complete discrete valuation ring of characteristic zero with algebraically closed residue field of prime characteristic p. Using the theory of Auslander–Reiten quivers, I prove that the obvious generalizations of the simple plane curve singularities do indeed have finite CM type. This uses a so-called Brauer–Thrall type theorem, which under certain hypotheses allows us to conclude that a connected component of an Auslander–Reiten quiver is the whole quiver. This Brauer–Thrall theorem places some restrictions on the residue field characteristic of R, and as a result our computations are incomplete in two cases.

Notations

Throughout, all rings will be commutative and Noetherian, and all modules will be finitely generated. A local ring $(R, \mathfrak{m}, k)$ is a ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. To say that a property holds on the punctured spectrum of a local ring $(R, \mathfrak{m})$ means that the property holds for all localizations $R_{\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Finally, we write R^{h} and $\widehat{R}$ for the Henselization and $\mathfrak{m}$-adic completion, respectively, of a local ring R.

We abbreviate the assertion "M is isomorphic to a direct summand of N" to "$M \mid N$". Denote by $\operatorname{syz}_R^n(M)$ the n^{th} syzygy in an arbitrary free resolution of an R-module M; it is unique up to projective direct summands.

Previous Results

The starting point for this dissertation is the paper of R. Wiegand [39]. Wiegand addresses a conjecture of Schreyer [33]: A local ring R has finite CM type if and only if the completion $\widehat{R}$ has finite CM type. Descent of finite CM type follows from the following theorem.

Theorem 0.1 ([39, Theorem 1.4])**.** *Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a flat local homomorphism. Let $\mathcal{C}(R)$ and $\mathcal{C}(S)$ be classes of finitely generated R-modules (resp. S-modules) which are closed under taking direct summands, and assume that $S \otimes_R M \in \mathcal{C}(S)$ whenever $M \in \mathcal{C}(R)$. If $\mathcal{C}(S)$ contains only finitely many indecomposable modules up to isomorphism, the same holds for $\mathcal{C}(R)$.*

Wiegand also isolates the property that guarantees ascent of finite CM type.

Lemma 0.2 ([39, Lemma 2.1])**.** *Let $R \longrightarrow S$ be a ring homomorphism with S semilocal, and let $\mathcal{C}(R)$ and $\mathcal{C}(S)$ be classes of finitely generated modules which are closed under taking direct summands. Assume that for every $M \in \mathcal{C}(S)$ there is some $X \in \mathcal{C}(R)$ such that M is isomorphic to a direct summand of $S \otimes_R X$ as an S-module. If $\mathcal{C}(R)$ contains only finitely many indecomposable modules up to isomorphism, the same holds for $\mathcal{C}(S)$.*

This criterion is used to prove ascent of finite CM type in many cases. We will repeatedly use the following result, part of the main theorem of [39], to establish ascent of finite CM type in greater generality.

Theorem 0.3 ([39, Theorem 2.9])**.** *Let $(R, \mathfrak{m})$ be a CM local ring which is Gorenstein on the punctured spectrum. If R has finite CM type, then the Henselization R^{h} has finite CM type. If in addition R^{h} is excellent, then the completion $\widehat{R}$ has finite CM type as well.*

We also include the following technical lemma, which will be useful in Chapters 1 and 2.

Lemma 0.4. *Let* $\varphi : (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ *be a flat local homomorphism of local rings. If* S *is regular (resp., Gorenstein) on the punctured spectrum, then* R *is as well. The converse holds if* φ *has regular (resp., Gorenstein) fibres.*

Proof. First assume that S is regular on the punctured spectrum and let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Since $R \longrightarrow S$ is a faithfully flat extension, the induced map $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ is surjective [26, 7.3], so there exists a prime ideal $\mathfrak{q}$ which contracts to $\mathfrak{p}$. Since $\mathfrak{n} \cap R = \mathfrak{m}$, we have that $\mathfrak{q} \neq \mathfrak{n}$. Thus $S_{\mathfrak{q}}$ is regular, and by [2, (2.2.12)], $R_{\mathfrak{p}}$ is regular. So R is an isolated singularity. For the converse, assume that R is an isolated singularity, that is, regular on the punctured spectrum. Let $\mathfrak{q} \in \operatorname{Spec}(S) \setminus \{\mathfrak{n}\}$, and set $\mathfrak{p} = \mathfrak{q} \cap R$. Since $R \longrightarrow S$ is flat, it satisfies the "going-down" property [26, Theorem 9.5]. Hence $\mathfrak{p} \neq \mathfrak{m}$, so $R_{\mathfrak{p}}$ is regular. The closed fibre of the flat local map $R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}$ is regular by hypothesis, so $S_{\mathfrak{q}}$ is regular by [2, (2.2.12)]. Since $\mathfrak{q}$ was an arbitrary nonmaximal prime, S is regular on the punctured spectrum.

For the statements about the Gorenstein property, we repeat the same argument. In this case, we have flat local homomorphisms $R_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}$, and $S_{\mathfrak{q}}$ is Gorenstein if and only if $R_{\mathfrak{p}}$ and the closed fibre are Gorenstein, by [2, (3.3.15)]. □

Chapter 1

Ascent for Excellent Rings

In this chapter I prove ascent of finite CM type for excellent CM local rings. That is, if R is an excellent CM local ring of finite CM type, then the $\mathfrak{m}$-adic completion $\widehat{R}$ also has finite CM type. It is convenient to introduce the Henselization R^{h} as an intermediate step. First I show that if R is a CM local ring of finite CM type, then R has *finite syzygy type*, which ascends easily to R^{h}. If R^{h} has a canonical module, basic properties of the canonical module show that this finite syzygy type implies that R^{h} is Gorenstein on the punctured spectrum (Corollary 1.2.4). Then R is Gorenstein on the punctured spectrum by Lemma 0.4, and Theorem 0.3 shows that the completion has finite CM type.

1.1 The Canonical Module and Lemmas

One of the main arguments of this chapter is that a certain ring is Gorenstein on the punctured spectrum The property that we will use to verify that a ring is Gorenstein on the punctured spectrum involves the canonical module ω_R. See [2, Chapter 3] for a full account of the canonical module; we reproduce here some of the basic properties we will use.

Let $(R, \mathfrak{m}, k)$ be a CM local ring of dimension d. A finitely generated R-module ω is a *canonical module* for R if

$$\dim_k \operatorname{Ext}^i_R(k, \omega) = \begin{cases} 0 & \text{if } i \neq d \\ 1 & \text{if } i = d. \end{cases}$$

Equivalently, ω is a MCM R-module of finite injective dimension and the natural map

$R \longrightarrow \mathrm{End}_R(\omega)$ is an isomorphism.

Not every CM local ring has a canonical module (see Chapter 2 and, for example, [32]). H.-B. Foxby [13] and I. Reiten [30] showed independently that a CM local ring R has a canonical module if and only if R is a homomorphic image of a Gorenstein local ring. In particular, a complete CM local ring is a homomorphic image of a regular local ring by Cohen's structure theorem [2, (A.21)], and so has a canonical module. Canonical modules respect localization and completion, and if ω is a canonical module and $x \in \mathfrak{m}$ is a nonzerodivisor, then x is ω-regular and $\omega/x\omega$ is a canonical module for R/xR [2, (3.3.5)]. When a CM local ring R does have a canonical module, it is unique up to isomorphism [2, (3.3.4)].

Canonical modules give a duality theory for CM modules. For a MCM R-module M, $\mathrm{Hom}_R(M, \omega)$ is again MCM, and $\mathrm{Hom}_R(\mathrm{Hom}_R(M, \omega), \omega) \cong M$ [2, (3.3.10)]. Further, $\mathrm{Ext}^i_R(M, \omega) = 0$ for $i > 0$. In particular, ω is an injective object in the category of MCM R-modules: if ω appears as the left-hand end of a short exact sequence of MCM modules, then the sequence splits. Finally, the ring R is Gorenstein if and only if $\omega \cong R$.

We also record the following lemma from [10]; note that this is not the precise statement that appears there, but is what is actually proved. We include the proof for completeness, and since the result will reappear in Chapter 2. Recall that a finitely generated module M over a Noetherian ring R is said to satisfy Serre's condition (S_k) if $\mathrm{depth}(M_\mathfrak{p}) \geq \min\{\mathrm{height}\,\mathfrak{p}, k\}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Lemma 1.1.1 ([10, Theorem 3.8]). *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d. Fix an integer $k \leq d$, and assume that $R_\mathfrak{p}$ is Gorenstein for every prime $\mathfrak{p}$ of height at most $k-1$. Then every R-module satisfying (S_k) is a k^{th} syzygy of some finitely generated R-module.*

Proof. Let M be an R-module satisfying (S_k). We may assume that $k \geq 3$ by Theorems 3.5 and 3.6 of [10]. Then M is reflexive by [10, Theorem 3.6]. Resolve $M^* := \mathrm{Hom}_R(M, R)$:

$$\cdots \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0 \tag{1.1.1}$$

and dualize, obtaining a complex

$$0 \longrightarrow M^{**} \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{k-1}^* \longrightarrow C \longrightarrow 0, \tag{1.1.2}$$

where $C := \operatorname{coker}(F_{k-2}^* \longrightarrow F_{k-1}^*)$. Since $M^{**} \cong M$, it will suffice to show that (1.1.2) is exact.

If (1.1.2) is not exact, choose the least index i, $1 \le i \le k-2$, such that the sequence fails to be exact at F_i^*. Let D be the cokernel of $F_{i-1}^* \longrightarrow F_i^*$. Then D contains a submodule isomorphic to $\operatorname{Ext}_R^i(M^*, R)$, which is nonzero by our choice of i. Choose $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{height}(\mathfrak{p}) = k$. Then for any $\mathfrak{q} \subset \mathfrak{p}$, $R_\mathfrak{q}$ is Gorenstein and M_q is a MCM $R_\mathfrak{q}$-module, so $\operatorname{Ext}_{R_\mathfrak{q}}^i(M_\mathfrak{q}^*, R_\mathfrak{q}) = 0$. Thus $\operatorname{Ext}_{R_\mathfrak{p}}^i(M_\mathfrak{p}^*, R_\mathfrak{p})$ is a nonzero $R_\mathfrak{p}$-module of finite length, so $\operatorname{depth} D_\mathfrak{p} = 0$. The depth lemma then implies that $\operatorname{depth} M_\mathfrak{p} = i+1 < k$, a contradiction. □

The next lemma isolates the property we will use to show that a CM local ring of finite CM type, with a canonical module, is Gorenstein on the punctured spectrum.

Lemma 1.1.2. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Then R is Gorenstein on the punctured spectrum if and only if ω is isomorphic to a direct summand of a d^{th} syzygy of some finitely generated R-module.*

Proof. If R is Gorenstein on the punctured spectrum, then every MCM R-module, in particular ω, is a d^{th} syzygy by Lemma 1.1.1. For the converse, let M be a finitely generated R-module such that $\omega \mid \operatorname{syz}_R^d(M)$. We have an exact sequence

$$0 \longrightarrow \operatorname{syz}_R^d(M) \xrightarrow{\alpha} F \longrightarrow \operatorname{syz}_R^{d-1}(M) \longrightarrow 0, \tag{1.1.3}$$

with F a finitely generated free R-module. Use the split surjection $\pi : \operatorname{syz}_R^d(M) \twoheadrightarrow \omega$ to form a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{syz}_R^d(M) & \xrightarrow{\alpha} & F & \longrightarrow & \operatorname{syz}_R^{d-1}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & \omega & \xrightarrow[\beta]{} & X & \longrightarrow & \operatorname{syz}_R^{d-1}(M) & \longrightarrow & 0. \end{array} \tag{1.1.4}$$

Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be a nonmaximal prime. Since $\omega_\mathfrak{p}$ is a canonical module for $R_\mathfrak{p}$ and $\operatorname{syz}_R^{d-1}(M)_\mathfrak{p}$ is a MCM $R_\mathfrak{p}$-module, the second row of (1.1.4) splits when localized at $\mathfrak{p}$. Let $\rho : X_\mathfrak{p} \longrightarrow \omega_\mathfrak{p}$ be a splitting map for β, so that $\rho\beta = 1_{\omega_\mathfrak{p}}$, and let $j : \omega_\mathfrak{p} \longrightarrow \operatorname{syz}_R^d(M)_\mathfrak{p}$ be a splitting map for π, so that $\pi j = 1_{\omega_\mathfrak{p}}$. Then $\rho f \alpha j = \rho\beta\pi j = 1_{\omega_\mathfrak{p}}$. This shows that $\omega_\mathfrak{p}$ is a direct summand of $F_\mathfrak{p}$, and so is free. Hence $R_\mathfrak{p}$ is Gorenstein, as desired. □

1.2 Ascent for Excellent Rings

Not every excellent CM local ring has a canonical module (see, for example, [32] and Section 2.4). Thus, in order to apply Lemma 1.1.2, we change base to a ring which is known to have a canonical module. Let R be an excellent CM local ring. Then the Henselization R^{h} of R is also excellent, by [14, Theorem 5.3]. It is a consequence of Néron–Popescu desingularization [38, Theorem 2.4] that an excellent Henselian local ring satisfies the Artin approximation property. V. Hinich showed in [20] that Artin approximation implies the existence of a canonical module (see also [31]).

For a local ring R of dimension d, we let $\mathcal{S}(R)$ be the class of all R-modules M such that M is isomorphic to a direct summand of a d^{th} syzygy of a finitely generated R-module. Note that since we do not require free resolutions to be minimal, finitely generated free R-modules are d^{th} syzygies.

Definition 1.2.1. *We say that a local ring* $(R, \mathfrak{m})$ *of dimension* d *has* finite syzygy type *provided there are, up to isomorphism, only finitely many indecomposable modules in* $\mathcal{S}(R)$.

It is clear that finite CM type implies finite syzygy type for CM local rings. See Corollary 1.2.4 for a partial converse. Just as important for our purposes, finite syzygy type ascends to the Henselization.

Proposition 1.2.2. *Let* $(R, \mathfrak{m})$ *be a local ring with Henselization* R^{h}. *Then* R *has finite syzygy type if and only if* R^{h} *has finite syzygy type.*

Proof. Descent follows from the general result of Wiegand reproduced in Chapter 0 as Theorem 0.1, since $R \longrightarrow R^{\mathrm{h}}$ is a flat local homomorphism. For ascent, it suffices by Lemma 0.2 to show that if an R^{h}-module M is a direct summand of a d^{th} syzygy over R^{h}, then $M \mid R^{\mathrm{h}} \otimes_R N$ for some $N \in \mathcal{S}(R)$.

Define $\mu : R^{\mathrm{h}} \otimes_R R^{\mathrm{h}} \longrightarrow R$ by $\mu(a \otimes b) = ab$, and let J be the kernel of μ. We claim that the exact sequence

$$0 \longrightarrow J \longrightarrow R^{\mathrm{h}} \otimes_R R^{\mathrm{h}} \xrightarrow{\mu} R^{\mathrm{h}} \longrightarrow 0 \tag{1.2.1}$$

splits as $R^{\mathrm{h}} \otimes_R R^{\mathrm{h}}$-modules. This property is referred to as *separability* in [6]. We know ([27, p.37]) that R^{h} is the directed union of pointed étale extensions $(S, \mathfrak{n})$ of R. For any such S, the map $R/\mathfrak{m} \longrightarrow S/\mathfrak{m}S$ is trivially étale (being an isomorphism!).

By [6, 7.1], each extension $R \longrightarrow S$ is separable, so each S is projective as an $S \otimes_R S$-module. It follows that R^{h} is flat, so projective over $R^{\mathrm{h}} \otimes_R R^{\mathrm{h}}$, whence (1.2.1) splits. This proves the claim.

Let $M \in \mathcal{S}(R^{\mathrm{h}})$, and let X be a finitely generated R^{h}-module such that M is a direct summand of $\operatorname{syz}^d_{R^{\mathrm{h}}}(X)$. Applying the functor $- \otimes_{R^{\mathrm{h}}} X$ to the sequence (1.2.1), we get a split exact sequence of R^{h}-modules

$$0 \longrightarrow J \otimes_R X \longrightarrow R^{\mathrm{h}} \otimes_R X \longrightarrow X \longrightarrow 0. \tag{1.2.2}$$

Thus X is a direct summand of the extended module $R^{\mathrm{h}} \otimes_R X$, where the action of R^{h} on $R^{\mathrm{h}} \otimes_R X$ is via change of rings. Write ${}_R X$ as a directed union of finitely generated R-modules Y_α. Then, since X is finitely generated as an R^{h}-module, $X \mid R^{\mathrm{h}} \otimes_R Y_\alpha$ for some Y_α. Set $Z = \operatorname{syz}^d_R(Y_\alpha)$. Then there exists a free module $(R^{\mathrm{h}})^n$ so that $M \mid (R^{\mathrm{h}} \otimes_R Z) \oplus (R^{\mathrm{h}})^n$ as R^{h}-modules. Put $N = Z \oplus R^n$ to finish the proof. □

Proposition 1.2.3. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Let $\underline{x} = \{x_1, \ldots, x_d\}$ be a system of parameters for R. For each integer $n \geq 1$, let $\underline{x}^n = \{x_1^n, \ldots, x_d^n\}$ and let Σ_n be the set of isomorphism classes of R-modules appearing in direct-sum decompositions of direct sums of copies of $\operatorname{syz}^d_R(\omega/(\underline{x}^n)\omega)$. Set $\Sigma := \bigcup_n \Sigma_n$. If Σ contains only finitely many isomorphism classes of indecomposables, then R is Gorenstein on the punctured spectrum.*

Proof. First note that if $d = 0$, the conclusion is vacuous, so we may safely assume $d \geq 1$. Fix an integer $n \geq 1$ for a moment, and set $\overline{\omega} = \omega/(\underline{x}^n)\omega$. Then we have an exact sequence

$$0 \longrightarrow \operatorname{syz}^d_R(\overline{\omega}) \longrightarrow F \longrightarrow \operatorname{syz}^{d-1}_R(\overline{\omega}) \longrightarrow 0 \tag{1.2.3}$$

with F a finitely generated free R-module. We apply the functor $(-)' := \operatorname{Hom}_R(-, \omega)$, and get an exact sequence

$$0 \longrightarrow (\operatorname{syz}^{d-1}_R(\overline{\omega}))' \longrightarrow F' \longrightarrow (\operatorname{syz}^d_R(\overline{\omega}))' \longrightarrow \operatorname{Ext}^1_R(\operatorname{syz}^{d-1}_R(\overline{\omega}), \omega) \longrightarrow 0. \tag{1.2.4}$$

Note that by [26, Lemma 18.2] and [2, (3.3.5)],

$$\operatorname{Ext}^1_R(\operatorname{syz}^{d-1}_R(\overline{\omega}), \omega) \cong \operatorname{Ext}^d_R(\overline{\omega}, \omega) \cong \operatorname{Hom}_R(\overline{\omega}, \overline{\omega}) \cong R/(\underline{x}^n), \tag{1.2.5}$$

so we get a surjection $(\operatorname{syz}^d_R(\overline{\omega}))' \longrightarrow R/(\underline{x}^n)$. Since R is local, there is an indecomposable direct summand X_n of $(\operatorname{syz}^d_R(\overline{\omega}))'$ mapping onto $R/(\underline{x}^n)$.

Now allow n to vary over all positive integers. The set of isomorphism classes $\{[X'_n]\}_{n\geq 1}$ is contained in Σ, so it is a finite set. Then $\{[X_n]\}_{n\geq 1}$ is also a finite set. Hence there exists an integer m such that the indecomposable module X_m maps onto $R/(\underline{x}^n)$ for infinitely many n. I claim that this forces X_m to be free. The surjections $X_m \longrightarrow R/(\underline{x}^n)$ induce surjections $X_m/\underline{x}^n X_m \longrightarrow R/(\underline{x}^n)$ for each n. Thus $R/(\underline{x}^n) \mid X_m/\underline{x}^n X_m$ for infinitely many n, and therefore *every* $n \geq 1$. It follows from [17, Cor. 2] (reproduced as Lemma 2.3.1 below) that $R \mid X_m$. Since X_m is indecomposable, this shows that X_m is free, so $(\mathrm{syz}_R^d(\omega/\underline{x}^m\omega))'$ has a nonzero free direct summand. Dualizing, we see that ω is a direct summand of $\mathrm{syz}_R^d(\omega/(\underline{x}^m)\omega)$. By Lemma 1.1.2, then, R is Gorenstein on the punctured spectrum. □

We record as a corollary the case of particular interest, from which the main theorem of this chapter will follow.

Corollary 1.2.4. *Let $(R, \mathfrak{m})$ be a CM local ring of dimension d with canonical module ω. Assume that R has finite syzygy type. Then R is Gorenstein on the punctured spectrum, and in particular has finite CM type.*

Proof. The first statement follows directly from Proposition 1.2.3. For the second, we apply Lemma 1.1.1. □

Theorem 1.2.5. *Let $(R, \mathfrak{m})$ be a CM local ring such that the Henselization R^{h} has a canonical module. If R has finite CM type, then R^{h} has finite CM type.*

Proof. If R has finite CM type, then in particular, R has finite syzygy type. By Proposition 1.2.2, finite syzygy type ascends to R^{h}. So R^{h} is Gorenstein on the punctured spectrum by Corollary 1.2.4. Finally, Lemma 1.1.1 implies that R^{h} has finite CM type. □

Recall that a Noetherian ring R is called *excellent* provided R is universally catenary, R has geometrically regular formal fibres, and every finitely generated R-algebra S has open regular locus. The main result of this chapter is that for this wide class of rings, finite CM type ascends to the completion.

Theorem 1.2.6. *Let $(R, \mathfrak{m})$ be a CM local ring such that the Henselization R^{h} is excellent. If R has finite CM type, then the Henselization R^{h} and the completion $\widehat{R}$ have finite CM type.*

Proof. By [38, Theorem 2.4], R^{h} satisfies the Artin Approximation property. This implies that R^{h} has a canonical module ([20], [31]). We now apply Theorem 1.2.5 to see that R^{h} has finite CM type. Theorem 0.3 shows that $\widehat{R}$ has finite CM type as well. □

Theorem 1.2.7. *Let $(R, \mathfrak{m})$ be an excellent CM local ring. Then R has finite Cohen–Macaulay type if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has finite Cohen–Macaulay type.*

Proof. Descent follows from Theorem 0.1. For the converse, suppose R has finite CM type. S. Greco [14, Theorem 5.3] shows that R^{h} is also excellent, and so Theorem 1.2.6 finishes the proof. □

We also extend the result of Auslander mentioned in the Introduction, that a complete CM local ring of finite CM type is regular on the punctured spectrum, to this more general situation.

Corollary 1.2.8. *Let $(R, \mathfrak{m})$ be an excellent CM local ring of finite CM type. Then R is an isolated singularity.*

Proof. By Theorem 1.2.7, the completion $\widehat{R}$ also has finite CM type. By Auslander's theorem for complete local rings [1], $\widehat{R}$ is an isolated singularity. This property satisfies faithfully flat descent (Lemma 0.4), so R is an isolated singularity as well. □

Chapter 2

Ascent with Gorenstein modules

In this chapter we show that finite CM type ascends from R to $\widehat{R}$ when R is a CM local ring with a Gorenstein module and the Henselization R^{h} is excellent. As in Chapter 1, the problem reduces to showing that if R has finite CM type and a Gorenstein module, then R is Gorenstein on the punctured spectrum.

2.1 Background on Gorenstein Modules

Here we collect some relevant facts concerning Gorenstein modules, which were introduced by R.Y. Sharp in [34] and studied extensively in [11], [13], [35], [36].

Definition 2.1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, and G a finitely generated R-module. We say that G is a* Gorenstein module for R of type r *provided*

$$\mu_R^i(\mathfrak{m}, G) = \begin{cases} 0 & \text{if } i \neq d \\ r & \text{if } i = d, \end{cases}$$

where $\mu_R^i(\mathfrak{m}, M) = \dim_k \operatorname{Ext}_R^i(k, M)$ is the i^{th} Bass number of M.

This definition is equivalent [2, (1.2.5), (3.1.14), (1.2.15)] to saying that G is a MCM R-module of finite injective dimension and type r. Also, existence of a Gorenstein R-module forces R to be CM ([35, (3.9)]). With this in mind, we see that a canonical module for R, if it exists, is precisely a Gorenstein module of type 1 (*cf.* Section 1.1). If R does have a canonical module ω, then every Gorenstein R-module is isomorphic to a direct sum of copies of ω ([11, (4.6)]). The completion $\widehat{G}$ of a

Gorenstein R-module G is a Gorenstein $\widehat{R}$-module, so we see that $\widehat{G} \cong \omega^r$, where r is the type of G and ω is the canonical module for $\widehat{R}$ (which exists, by Cohen's structure theorem [2, (A.21)]).

As might be expected, Gorenstein modules enjoy a number of useful homological properties. Most of these can be seen by passing to the completion and using known properties of canonical modules as in Section 1.1, but we provide references here for completeness. Let R be a CM local ring with Gorenstein module G of type r. The endomorphism ring $\mathrm{Hom}_R(G, G)$ is a free R-module of rank r^2, and $\mathrm{Ext}^i_R(G, G) = 0$ for $i > 0$ ([12, (3.1)]). For a MCM R-module M, $\mathrm{Hom}_R(M, G)$ is again MCM, and $\mathrm{Hom}_R(\mathrm{Hom}_R(M, G), G) \cong M^{r^2}$ ([36, (2.10)]). Finally, if $x \in \mathfrak{m}$ is an R-regular element, then x is also G-regular, and G/xG is a Gorenstein R/xR-module of the same type r, with $\mathrm{Hom}_R(G, G)/x\,\mathrm{Hom}_R(G, G) \cong \mathrm{Hom}_{R/xR}(G/xG, G/xG)$ ([35, (4.13),(4.11)]).

2.2 Gorenstein modules and Finite Index

Let $(R, \mathfrak{m})$ be a local ring. Following S. Ding [7], we define index(R) to be the least positive integer n so that $R/\mathfrak{m}^n$ is not a homomorphic image of a MCM R-module without free summand. If no such n exists, say that index$(R) = \infty$. Note that index$(R) = 1$ if and only if R is a regular local ring. Our goal in this section is to prove the following.

Theorem 2.2.1. *Let R be a CM local ring of dimension d. Assume that R has a Gorenstein module G. Then the following conditions are equivalent:*

1. *The ring R has finite index;*
2. *The ring R is Gorenstein on the punctured spectrum, that is, $R_{\mathfrak{p}}$ is Gorenstein for all primes $\mathfrak{p} \neq \mathfrak{m}$;*
3. *There exists an R-regular sequence $x_1, \ldots, x_d$ such that $R/(x_1, \ldots, x_d)$ is not a homomorphic image of a MCM R-module with no free summands;*
4. *Every MCM R-module is a d^{th} syzygy.*

In particular, this recovers a result of Ding for a CM local ring R with a canonical module: index$(R) < \infty$ if and only if R is Gorenstein on the punctured spectrum [7]. Note also that the implication (2) $\implies$ (4) is a special case of Lemma 1.1.1.

Let R be a Noetherian ring and let G be a finitely generated R-module. We have a natural homomorphism

$$\alpha_G : \operatorname{Hom}_R(G,R) \otimes_R G \longrightarrow \operatorname{Hom}_R(G,G) \tag{2.2.1}$$

given by $\alpha_G(f \otimes g)(g') = f(g')g$.

Lemma 2.2.2. *Let $P(G,G)$ be the set of all endomorphisms $h : G \longrightarrow G$ that factor through a free R-module. Then $\operatorname{Im} \alpha_G = P(G,G)$.*

Proof. Let $h \in \operatorname{Im} \alpha_G$, so that there exist $f_i : G \longrightarrow R$ and $x_i \in G$, $i = 1, \dots, n$, such that $h(y) = \sum_1^n f_i(y) x_i$ for all $y \in G$. Then h factors through R^n, for if we let $g = [x_1, \cdots, x_n]$, and $f = [f_1 \cdots f_n]^t$, then $h = gf : G \longrightarrow R^n \longrightarrow G$.

For the other inclusion, suppose we have $h : G \longrightarrow G$ factoring through R^n as $h = gf$. Let $g_1, \dots, g_n$ be the restrictions of g to each copy of R. Let f_j be the composition of f with the projection to the j^{th} component of R^n. Then we have $h = \sum_{j=1}^n g_j f_j : G \longrightarrow G$. Let $x_i = g_i(1)$. Then for any $y \in G$

$$h(y) = \sum_{j=1}^{n} f_i(y) x_i = \alpha_G\big(\sum_{j=1}^{n} f_i \otimes x_i\big)(y).$$

This gives $h \in \operatorname{Im} \alpha_G$, as desired. □

• For the next four results, assume that $(R, \mathfrak{m})$ is a CM local ring and G is a Gorenstein R-module of minimum type r.

We may consider R as a submodule of $\operatorname{Hom}_R(G,G)$ via the map sending x to "multiplication by x". As G is a faithful R-module ([34, 4.2]), this map is injective. Define another submodule τ of $\operatorname{Hom}_R(G,G)$ by $\tau = R \cap \operatorname{Im} \alpha_G$. Then τ can be considered either as a submodule of $\operatorname{Hom}_R(G,G)$ or as an ideal of R. We point out that in the case $r = 1$, τ coincides with the trace of G in R. In general the two are distinct. However, τ does share the following important property with the trace.

Lemma 2.2.3. *The ring R is Gorenstein if and only if $\tau = R$.*

Proof. If R is Gorenstein, then since G has minimum type, $G \cong R$, and every endomorphism of G is given by multiplication by some element of R. Conversely, if $\tau = R$, then in particular the identity endomorphism of G is in τ. That is, G is isomorphic to a direct summand of a free module (Lemma 2.2.2), hence free. Therefore R is a direct summand of G and so is a Gorenstein ring. □

This gives the following fundamental observation:

Proposition 2.2.4. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $R_{\mathfrak{p}}$ is Gorenstein if and only if $\tau R_{\mathfrak{p}} = R_{\mathfrak{p}}$. In particular, R is Gorenstein on the punctured spectrum if and only if τ is an $\mathfrak{m}$-primary ideal.*

We now consider the functor on R-modules given by $M^{\vee} = \operatorname{Hom}_R(M, G)$. Similarly, if R is known to have a canonical module ω, we will write M' for the canonical dual $\operatorname{Hom}_R(M, \omega)$ of M.

Note: We have $\widehat{M^{\vee}} \cong \left(\left(\widehat{M}\right)'\right)^r$ for every finitely generated R-module M. Indeed, passing to the completion $\widehat{R}$ and letting $\omega = \omega_{\widehat{R}}$, we have

$$\begin{aligned} \widehat{M^{\vee}} &= \operatorname{Hom}_R(M, G) \otimes_R \widehat{R} \\ &= \operatorname{Hom}_{\widehat{R}}(\widehat{M}, \omega^r) \\ &= \operatorname{Hom}_{\widehat{R}}(\widehat{M}, \omega)^r. \end{aligned} \tag{2.2.2}$$

Let $x \in \mathfrak{m}$ be an R-regular, and so G-regular, element, and set $\overline{G} = G/xG$. Note that $Z := \operatorname{syz}^1_R(\overline{G})$ is a MCM R-module; in particular, we have $\widehat{Z}'' \cong \widehat{Z}$.

The next lemma, with $G = \omega$, is [7, Lemma 1.6]. Our proof is by reduction to that case. Recall that the notation $M \mid N$ means that M is isomorphic to a direct summand of N.

Lemma 2.2.5. *With notation as above, $(\operatorname{syz}^1_R(\overline{G}))^{\vee}$ has a nonzero free direct summand if and only if $x \in \tau$.*

Proof. Put $Z = \operatorname{syz}^1_R(\overline{G})$, and suppose that $R \mid Z^{\vee}$. Then $\widehat{R} \mid (\widehat{Z}')^r$. By the Krull–Schmidt uniqueness theorem for $\widehat{R}$-modules, $\widehat{R} \mid \widehat{Z}'$. Let ω be the canonical module for $\widehat{R}$ (which exists, by [2, (A.21)]). Dualizing into ω, we see that $\omega \mid \widehat{Z}$. But $\widehat{Z} \cong \operatorname{syz}_{\widehat{R}}(\omega^r/x\omega^r) \cong \operatorname{syz}_{\widehat{R}}(\overline{\omega})^r$, where $\overline{\omega} := \omega/x\omega$. Another application of the Krull–Schmidt theorem shows that $\omega \mid \operatorname{syz}_{\widehat{R}}(\overline{\omega})$. By [7, (1.6)], then, the map on ω given by multiplication by x factors through a free $\widehat{R}$-module. Since $\widehat{G} \cong \omega^r$, the corresponding map on $\widehat{G}$ also factors through a free $\widehat{R}$-module. Hence, by Lemma 2.2.2, $x \in \tau\widehat{R}$, and so $x \in \tau$.

For the other implication, suppose $x \in \tau$. Then $x \in \tau\widehat{R}$, so $\operatorname{syz}^1_{\widehat{R}}(\overline{\omega})$ has a direct summand isomorphic to ω by [7, (1.6)]. Then $\operatorname{syz}^1_R(\overline{G})$ has a direct summand isomorphic to G, and $\operatorname{syz}^1_R(\overline{G}))(^{\vee}$ has a free summand. □

Proof of Theorem 2.2.1. If $d = 0$, there is nothing to prove, so assume $d > 0$. We begin with a construction which will be used in the proof.

Let $x \in \mathfrak{m}$ be an arbitrary R-regular element, and let

$$0 \longrightarrow \operatorname{syz}^1_R(\overline{G}) \longrightarrow R^m \longrightarrow \overline{G} \longrightarrow 0 \tag{2.2.3}$$

be the first part of a minimal free resolution of $\overline{G} := G/xG$. Then applying the functor $\operatorname{Hom}_R(-, G)$ gives an exact sequence

$$0 \longrightarrow \overline{G}^{\vee} \longrightarrow G^m \longrightarrow (\operatorname{syz}^1_R(\overline{G}))^{\vee} \longrightarrow \operatorname{Ext}^1_R(\overline{G}, G) \longrightarrow 0. \tag{2.2.4}$$

Note that $\overline{G}^{\vee} = 0$ since $\overline{G}$ has depth $d - 1$. We claim that $\operatorname{Ext}^1_R(\overline{G}, G) \cong \overline{R}^{r^2}$, where $\overline{R} := R/xR$. In fact, as $x\overline{G} = 0$ and x is G-regular, $\operatorname{Ext}^1_R(\overline{G}, G) \cong \operatorname{Hom}_{\overline{R}}(\overline{G}, \overline{G})$ ([2, (1.2.4)]). As pointed out in Section 2.1, $\overline{G}$ is a Gorenstein module for $\overline{R}$ of type r, and so $\operatorname{Hom}_{\overline{R}}(\overline{G}, \overline{G}) \cong \overline{R}^{r^2}$, as claimed. Since $\operatorname{syz}^1_R(\overline{G})$ is a MCM R-module, its G-dual $(\operatorname{syz}^1_R(\overline{G}))^{\vee}$ is also MCM. This gives the short exact sequence

$$0 \longrightarrow G^m \longrightarrow (\operatorname{syz}^1_R(\overline{G}))^{\vee} \longrightarrow \overline{R}^{r^2} \longrightarrow 0, \tag{2.2.5}$$

with the middle term a MCM R-module.

1. $\Longrightarrow$ 2. We assume that R is not Gorenstein on the punctured spectrum. Fix an arbitrary positive integer n. Since the ideal τ is not $\mathfrak{m}$-primary (Proposition 2.2.4), $\mathfrak{m}^n \not\subseteq \tau$. Choose a regular element $x \in \mathfrak{m}^n - \tau$. Then Lemma 2.2.5 implies that $\operatorname{syz}^1_R(\overline{G})^{\vee}$ has no nonzero free summands. So in (2.2.5), we have constructed a MCM R-module with no nonzero free summands which maps onto R/xR. In turn, R/xR maps onto $R/\mathfrak{m}^n$ since $x \in \mathfrak{m}^n$. As n was arbitrary, this shows that $\operatorname{index}(R)$ is infinite.

2. $\Longrightarrow$ 3. By Proposition 2.2.4, R is Gorenstein on the punctured spectrum if and only if τ is $\mathfrak{m}$-primary. Thus there exists an R-sequence $x_1, \ldots, x_d$ in τ. We use induction on d to show that this is the desired R-sequence.

For the case $d = 1$, set $x = x_1$ and $\overline{R} = R/xR$. As before, we obtain the short exact sequence (2.2.5). By Lemma 2.2.5, since $x \in \tau$, $\operatorname{syz}^1_R(\overline{G})^{\vee} \cong U \oplus R$ for some R-module U. Denote the map $U \oplus R \longrightarrow \overline{R}^{r^2}$ in (2.2.5) by f. We claim first that $f(U) \neq \overline{R}^{r^2}$. If $f(U) = \overline{R}^{r^2}$, we have the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & \longrightarrow & f(U) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^m & \longrightarrow & U \oplus R & \longrightarrow & \overline{R}^{r^2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
\tag{2.2.6}
$$

By the Snake Lemma, then, $G^m \longrightarrow R$ is surjective, and so G is free and R is Gorenstein. In this case, $G \cong R$ and $U = 0$, a contradiction.

Now suppose that $g : Z \longrightarrow \overline{R}$ is a surjection with Z a MCM R-module. This gives a surjection $Z^{r^2} \longrightarrow \overline{R}^{r^2}$, which we also call g. Since $\mathrm{Ext}^1_R(Z^{r^2}, G^m) = 0$, g lifts to $h : Z^{r^2} \longrightarrow U \oplus R$ such that $g = fh$. We claim that the composition $\pi h : Z^{r^2} \longrightarrow U \oplus R \longrightarrow R$ is surjective. Suppose $\pi h(Z^{r^2}) \subseteq \mathfrak{m}$, and let $\alpha \in \overline{R}^{r^2}$. Write $\alpha = g(z) = f(h(z))$ for some $z \in Z^{r^2}$. Also write $h(z) = (u, r) \in U \oplus R$. Then $r \in \mathfrak{m}$ by assumption, so $\alpha = rf(0,1) + f(u,0) \in \mathfrak{m}\overline{R}^{r^2} + f(U)$. Since α was arbitrary, this shows that $\overline{R}^{r^2} = \mathfrak{m}\overline{R}^{r^2} + f(U)$, and Nakayama's lemma implies $f(U) = \overline{R}^{r^2}$, a contradiction. The surjection $Z^{r^2} \longrightarrow R$ shows that Z has a free summand, as desired.

Now suppose $d > 1$ and there exists a surjection $Z \longrightarrow R/(x_1, \ldots, x_d)$ with Z MCM. Then $\overline{Z} \longrightarrow \overline{R}/(\overline{x_2}, \ldots, \overline{x_d})$ is also a surjection, where a bar indicates reduction modulo x_1. Since $\overline{x_2}, \ldots, \overline{x_d}$ are in the extended ideal $\overline{\tau} = \tau\overline{R}$, $\overline{Z}$ has an $\overline{R}$-summand by the case $d = 1$. But then $Z \longrightarrow \overline{R}$ is surjective and applying the case $d = 1$ again shows that Z has a free summand.

3. $\implies$ 1. The ideal $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary, so $\mathfrak{m}^n \subseteq (x_1, \ldots, x_d)$ for some $n > 1$. Then the surjection $R/\mathfrak{m}^n \longrightarrow R/(x_1, \ldots, x_d)$ shows that no MCM R-module without a nonzero free summand maps onto $R/\mathfrak{m}^n$, that is, $\mathrm{index}(R) < \infty$.

2. $\implies$ 4. This is a reproduction of Lemma 1.1.1.

4. $\implies$ 2. Since every MCM R-module is a d^{th} syzygy, there exists an exact sequence

$$0 \longrightarrow G \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0 \tag{2.2.7}$$

with each F_i a finitely generated free R-module. Let M be the cokernel of $G \longrightarrow F_d$, the $(d-1)^{\text{th}}$ syzygy. Let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal, and localize at $\mathfrak{p}$. Then $M_{\mathfrak{p}}$ is a $(d-1)^{\text{th}}$ syzygy over the CM ring $R_{\mathfrak{p}}$, which has dimension $\leq d-1$, so $M_{\mathfrak{p}}$ is MCM by the depth lemma. Since $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, G_{\mathfrak{p}}) = 0$, the sequence $0 \longrightarrow G_{\mathfrak{p}} \longrightarrow (F_d)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0$ splits. Hence $G_{\mathfrak{p}}$ is free, and $R_{\mathfrak{p}}$ is Gorenstein. □

Note: The implication 2. $\implies$ 4. does not require the existence of a canonical or Gorenstein module. It would be interesting to find a proof of 4. $\implies$ 2. that also avoided Gorenstein modules.

2.3 Finite Index and Finite Cohen–Macaulay Type

In this short section, we apply a result of R. Guralnick (reproduced for convenience) to show that a CM local ring of finite CM type has finite index. This lets us prove that a CM local ring of finite CM type with a Gorenstein module is Gorenstein on the punctured spectrum. Then Theorem 0.3 shows that R^{h} has finite CM type as well. If R^{h} is excellent, then finite CM type ascends all the way to $\widehat{R}$. As a corollary, we again extend the result of Auslander [1] and see that a CM local ring of finite CM type with a Gorenstein module is an isolated singularity.

Lemma 2.3.1 ([17, Cor. 2])**.** *Let $(R, \mathfrak{m})$ be a local ring and let M and N be finitely generated R-modules. If $N/\mathfrak{m}^n N$ is isomorphic to a direct summand of $M/\mathfrak{m}^n M$ for every $n \gg 0$, then N is isomorphic to a direct summand of M.*

Theorem 2.3.2. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type. Then R has finite index.*

Proof. Let $\{M_1, \ldots, M_r\}$ be a complete set of representatives for the isomorphism classes of nonfree indecomposable MCM R-modules. Since no M_i has a nonzero free summand, there exist integers n_i, $1 \leq i \leq r$, so that for $s \geq n_i$ there exists no surjection $M_i \longrightarrow R/\mathfrak{m}^s$ (Lemma 2.3.1). Set $N = \max\{n_i\}$. Let X be any MCM R-module without nonzero free summands, and decompose $X = M_1^{a_1} \oplus \cdots \oplus M_r^{a_r}$. If there were a surjection $X \longrightarrow R/\mathfrak{m}^N$, then, since R is local, one of the summands $M_i^{a_i}$ would map onto $R/\mathfrak{m}^N$, contradicting the choice of N. As X was arbitrary, this shows that $\operatorname{index}(R) \leq N < \infty$. □

Corollary 2.3.3. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type. Assume that R has a Gorenstein module G. Then R is Gorenstein on the punctured spectrum.*

Proof. By Theorem 2.3.2, R has finite index. By Theorem 2.2.1, then, R is Gorenstein on the punctured spectrum. □

Theorem 2.3.4. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type with a Gorenstein module. Assume that the Henselization R^{h} is excellent. Then the completion $\widehat{R}$ has finite CM type.*

Proof. By Corollary 2.3.3, R is Gorenstein on the punctured spectrum, and Theorem 0.3 finishes the proof. □

Corollary 2.3.5. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type with a Gorenstein module. Then R is an isolated singularity.*

Proof. By Auslander's theorem mentioned in the Introduction, the Henselization (which has finite CM type by Theorem 0.3) has an isolated singularity. By Lemma 0.4, this descends to R. □

2.4 Examples

In this section we give a few examples demonstrating the sharpness of the results in Chapters 1 and 2. The first two show that Schreyer's conjecture can fail for local rings that are not CM; the third is the promised example, due to Nishimura, of an excellent CM local ring with no Gorenstein module.

It follows from the descent criterion of Theorem 0.1 that finite CM type descends to a local ring R from its completion, regardless of whether R is CM. The analogous statement for ascent is false, as the following two examples show. While two may seem like overkill, they fail for different enough reasons that it seems instructive to include both.

Example 2.4.1. Let $T = k[[x, y, z]]/(x^3 - y^7) \cap (y, z)$, where k is any field. Then T has infinite CM type. To see this, first set $R = k[[x, y]]/(x^3 - y^7)$. Then $R \cong k[[t^3, t^7]]$ has infinite CM type by the classification in [4]. Further, $R[[z]]$ has infinite CM type: the map $R \longrightarrow R[[z]]$ is flat [26, p.53] with CM closed fiber, and Theorem 1.1 applies. Now, $R[[z]] \cong T/(x^3 - y^7)$. It is clear that any two nonisomorphic $R[[z]]$-modules are nonisomorphic as T-modules, so it remains only to see that a MCM $R[[z]]$-module also has depth 2 when viewed as a T-module. This follows from [2, (1.2.26)], so T has infinite CM type.

It is easy to check that the image of x is a nonzerodivisor in T. By [24, Theorem 1], then, T is the completion of some local integral domain A. Then A has finite CM type; in fact, it has no MCM modules at all. For if A had a nonzero MCM module, then A would be universally catenary [21, §1]. This implies ([26, p. 252]) that A is formally equidimensional, that is, all minimal primes of T have the same dimension. This is clearly absurd.

Example 2.4.2. Let k be any field, and let $K = k(t_1, t_2, \ldots)$ be an extension of k of infinite transcendence degree. Let f be an irreducible polynomial in n variables over K so that $R_1 = K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/(f)$ has infinite CM type, and let g be an irreducible polynomial in m variables (with $m \neq n$) so that $R_2 = K[y_1, \ldots, y_m]_{(y_1,\ldots,y_m)}/(g)$ has infinite CM type. Set

$$R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f, g).$$

Semilocalize R by inverting the multiplicative set given by the complement of the union of the two ideals $(\underline{x}) = (x_1, \ldots, x_n)$ and $(\underline{y}) = (y_1, \ldots, y_m)$. Note that these two maximal ideals have different heights. By [5], there exists a subring A of R so that A is a local domain with maximal ideal $(\underline{x}) \cap (\underline{y})$, and $A \hookrightarrow R$ is a finite birational extension. Specifically, note that the two residue fields of R are isomorphic, both being purely transcendental extensions of K. Let ϵ_1 be the surjection $R \longrightarrow K$ with kernel $(\underline{x})$, and let ϵ_2 be the surjection $R \longrightarrow K$ with kernel $(\underline{y})$. Then

$$A = \{f \in R \mid \epsilon_1(f) = \epsilon_2(f)\}.$$

The construction in [5] shows that A fails to be catenary: every nonmaximal prime of A has exactly one prime of R lying over it, and the maximal ideal of A is precisely the intersection of the two maximal ideals of R. The preimages of the two saturated chains $0 \subseteq (x_1) \subseteq (x_1, x_2) \subseteq \cdots \subseteq (x_1, \ldots, x_n)$ and $0 \subseteq (y_1) \subseteq (y_1, y_2) \subseteq \cdots \subseteq (y_1, \ldots, y_m)$ give saturated chains of different lengths of primes in A from 0 to the maximal ideal. Thus A fails to be catenary, so has no nonzero MCM modules ([21, §1]). In particular, A has finite CM type. We will show that $\widehat{A}$ has infinite CM type. Let $\widehat{R}$ be the completion of R with respect to the Jacobson radical $(\underline{x}) \cap (\underline{y})$. Since the Jacobson radical of R is equal to the maximal ideal of A, we have $\widehat{R} \cong R \otimes_A \widehat{A}$. Since $A \longrightarrow R$ is birational and finite, there exists a nonzerodivisor $t \in R$ such that $tR \subseteq A$. Then we also have $t\widehat{R} \subseteq \widehat{A}$, so $\widehat{A} \longrightarrow \widehat{R}$ is also birational and finite. We claim that $\widehat{A}$ has infinite CM type. We can write $\widehat{R} \cong \widehat{R}_1 \times \widehat{R}_2$, so $\widehat{R}$ has infinite CM type (by Theorem 0.1). If two torsion-free $\widehat{R}$-modules are isomorphic as $\widehat{A}$-modules, we can use the birationality to clear denominators and get an $\widehat{R}$-isomorphism. So $\widehat{A}$ has infinite CM type, while A has no MCM modules.

The next example is due to J.-I. Nishimura [28]. His construction is quite complicated, so we do not give many details, only the relevant ring.

Example 2.4.3. Let K_0 be a countable field of characteristic zero and let K be a purely transcendental extension of K_0 of countable transcendence degree. Let

$$T = K[[X_1, X_2, X_3, X_4, X_5]]/(X_1X_5 - X_2X_4, X_1X_2 - X_3X_4, X_2^2 - X_3X_5) \quad (2.4.1)$$

Note that T is a complete three-dimensional non-Gorenstein CM normal domain such that the divisor class group $\mathrm{Cl}(T)$ is infinite cyclic. By [28, (5.6)], there exists an excellent three-dimensional factorial local domain $(A, \mathfrak{m})$ such that $\widehat{A} \cong T$. Suppose that A has a Gorenstein module G. Since A is factorial, $[G] = 0$ in $\mathrm{Cl}(A)$. Then $[\widehat{G}] = [(\omega_T)^r] = 0$ in $\mathrm{Cl}(T)$, where $r = \text{type}\,(G)$, so $r[\omega_T] = 0$, a contradiction. Hence A has no Gorenstein module.

Note also that the Henselization of A has a canonical module. Since T is Gorenstein on the punctured spectrum, ω_T is free on the punctured spectrum of T, so by [9, Théorème 3], is extended from the Henselization A^{h}. That is, A^{h} has a canonical module. This example shows that Theorem 1.2.7 is not just a special case of Theorem 2.3.4.

Chapter 3

Mixed Characteristic Hypersurfaces of Finite CM Type

This chapter is concerned with showing certain examples of complete hypersurfaces have finite CM type. To do this, we compute the Auslander–Reiten quivers of these rings. The AR quiver encapsulates much of the structure of the category of MCM modules over the ring. This structure is given in terms of Auslander–Reiten sequences, also known as almost split sequences.

• Throughout this chapter, R is a complete CM local ring with algebraically closed residue field k. Let M be an indecomposable MCM R-module. An *Auslander–Reiten (AR) sequence ending in M* is a nonsplit short exact sequence

$$0 \longrightarrow N \xrightarrow{p} E \xrightarrow{q} M \longrightarrow 0 \qquad (3.0.1)$$

such that N is an indecomposable MCM R-module, and any homomorphism of MCM R-modules $L \longrightarrow M$ that is not a split surjection factors through q. AR sequences are unique up to isomorphism of exact sequences when they exist. We say also that (3.0.1) is an *AR sequence starting from N*. A significant result of Auslander gives a necessary and sufficient condition for the existence of AR sequences.

Theorem 3.0.1 ([1]). *The ring R admits AR sequences (that is, for every nonfree indecomposable MCM R-module M there is an AR sequence ending in M) if and only if R is an isolated singularity.*

In the AR sequence (3.0.1), N is called the Auslander translation of M, and we write $N = \tau(M)$. The fact that R is a complete local ring implies that R has a canonical module, which gives a duality in the category of MCM R-modules. Thus R is an isolated singularity if and only if, for each indecomposable MCM module N not isomorphic to the canonical module, there is an AR sequence starting from N. It follows that $\tau(\tau(M)) \cong M$ [41, 2.14].

Closely related to AR sequences are irreducible homomorphisms. A homomorphism of MCM R-modules $\varphi : M \longrightarrow N$ is *irreducible* provided (1) φ is neither a split injection nor a split surjection, and (2) if φ factors through a MCM module X

$$\begin{array}{ccc} M & \xrightarrow{\quad\varphi\quad} & N \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & X & \end{array} \tag{3.0.2}$$

then either α is a split injection or β is a split surjection.

The next lemma follows from [41, 2.12] and the proof there.

Lemma 3.0.2 ([41, 2.12])**.** *Let M and L be indecomposable MCM R-modules, and assume that there exists an AR sequence (3.0.1) ending in M. The following conditions are equivalent.*

1. *L is isomorphic to a direct summand of E.*

2. *There is an irreducible homomorphism $L \longrightarrow M$.*

Each of these implies that the composition $L \longrightarrow E \longrightarrow M$ is an irreducible homomorphism.

In fact, the irreducible homomorphisms $L \longrightarrow M$ form a finite-dimensional k-vector space, and [41, 5.5] the dimension of this vector space is equal to the number of copies of L in the direct-sum decomposition of E.

For an indecomposable MCM R-module M, we encode all the above data into a graph.

Definition 3.0.3. *Assume R is an isolated singularity. The AR quiver Γ of R is a graph consisting of vertices, arrows, and dotted lines. The vertices are the isomorphism classes of indecomposable MCM R-modules. We draw n arrows $[A] \longrightarrow [B]$ if and only the dimension of the k-vector space of irreducible homomorphisms $A \longrightarrow B$ is n. We draw a dotted line between $[A]$ and $[B]$ if $A \cong \tau(B)$.*

The Auslander translation of M is easily calculated. Recall that the Auslander transpose $\operatorname{tr}(M)$ of M is $\operatorname{coker}\Phi^*$, where Φ is a presentation matrix for M.

Lemma 3.0.4 ([41, 3.13]). *Let $d = \dim(R)$. Then $\tau(M) = (\operatorname{syz}_R^d \operatorname{tr}(M))'$, where $(-)' = \operatorname{Hom}_R(-, \omega)$.*

The rest of this chapter is devoted to showing that certain complete hypersurfaces have finite CM type. The complete hypersurfaces containing a field and having finite CM type have been completely classified (see Chapter 0), so I will consider hypersurfaces not containing a field. This introduces some technical difficulties, especially in proving that a given connected component of an AR quiver is the entire quiver. This is the focus of Section 3.1.

The following lemma will be essential to understanding the structure of the AR quiver.

Lemma 3.0.5 ([41, 5.9]). *Assume R (complete with algebraically closed residue field) is an isolated singularity. Then the AR quiver Γ of R is a locally finite graph (that is, each vertex of Γ has only finitely many arrows starting from it or ending in it).*

3.1 A Brauer–Thrall Theorem in Mixed Characteristic

In proving that certain complete equicharacteristic hypersurfaces have finite Cohen–Macaulay type, Yoshino (like Buchweitz, Greuel and Schreyer [3]) uses the following theorem [41, 6.2]:

Theorem 3.1.1. *Assume R (as above) is an isolated singularity and that R contains a field. Let Γ be the Auslander–Reiten quiver of R, and assume that Γ^0 is a connected component of Γ with bounded multiplicity type. Then $\Gamma^0 = \Gamma$ and Γ is a finite graph. In particular, R has finite CM type.*

In the statement of Theorem 3.1.1, to say that Γ^0 has bounded multiplicity type means that there exists a such that for any $[M]$ in Γ^0, $e(M) < a$. Recall that the multiplicity $e(M)$ of a module M over a local ring R is $d!$ times the leading coefficient of the Hilbert polynomial of M, where $d = \dim(R)$. If R is an integral domain (or, more generally, if M is free of constant rank at the associated primes of R), then $e(M) = e(R)\dot{\operatorname{rank}}(R)$.

All issues of connectedness in Γ refer to the undirected graph obtained by replacing each arrow by an undirected edge and ignoring the dotted lines.

Theorem 3.1.1 is called a Brauer–Thrall type theorem in [40], by analogy with the First Brauer–Thrall Theorem in the representation theory of Artin algebras. We would like to use a result of this form to help us classify the mixed characteristic hypersurfaces of finite CM type. See Theorem 3.1.8.

• The following notations will be in effect for the rest of this section. Let V be a complete discrete valuation ring of characteristic zero with uniformizing parameter π and algebraically closed residue field of characteristic $p > 0$. There will be certain restrictions on p in what follows. Let R be a one-dimensional hypersurface over V, that is, $R = V[[y]]/(f(y))$ for some non-zero power series f in the maximal ideal of $V[[y]]$. We assume that π is not a factor of f, that is, π is a nonzerodivisor in R.

Note first that we may assume that f is a monic polynomial in y with coefficients in V. Write $f = \sum_{n=0}^{\infty} u_n \pi^{a_n} y^n$, where the a_n are nonnegative integers and u_n are units of V. Since π does not divide f by assumption, $a_n = 0$ for some $n > 0$. Let m be the smallest integer such that $a_m = 0$. Then f is *regular of order* m (see [42]). By the Weierstrass Preparation Theorem ([23, IV, 9.2]), there is a linear change of variable, σ, such that $R \cong V[[y]]/(\sigma(f))$ and $\sigma(f)$ is a monic polynomial of degree m, in which the coefficient of y^i is divisible by π for each $i < m$.

It follows from [23, IV, 9.1] that R is a finitely generated V-module, generated by the powers of y, $\{1, y, \ldots, y^{m-1}\}$.

Recall that the *Noether different* $N_V(R)$ of R over V is defined as follows: let $\mu : R \otimes_V R \longrightarrow R$ be the multiplication map, and let J be the kernel, so we have the exact sequence

$$0 \longrightarrow J \longrightarrow R \otimes_V R \xrightarrow{\mu} R \longrightarrow 0. \qquad (3.1.1)$$

Set $N_V(R) = \mu(\mathrm{Ann}_{R\otimes_V R}(J))$.

Lemma 3.1.2. *With notation as above,* $f'(y) \in N_V(R)$.

Proof. Write $f(y) = \sum_{i=0}^{n} v_i y^i$, where $v_i \in V$. Then $f'(y) = \sum_{i=1}^{n} i v_i y^{i-1}$. Put

$$\alpha = \sum_{i=1}^{n} \sum_{j=0}^{i-1} v_i (y^j \otimes y^{i-j-1}).$$

It is easy to check that $\mu(\alpha) = f'$. I claim that $\alpha \in \mathrm{Ann}_{R\otimes_V R}(J)$. First note that

$\alpha(1 \otimes y - y \otimes 1) = 0$:

$$\begin{aligned}
\alpha(1 \otimes y - y \otimes 1) &= \alpha(1 \otimes y) - \alpha(y \otimes 1) \\
&= \sum_{i=1}^{n} \sum_{j=0}^{i-1} v_i(y^j \otimes y^{i-j}) - \sum_{i=1}^{n} \sum_{j=0}^{i-1} v_i(y^{j+1} \otimes y^{i-j-1}) \\
&= \sum_{i=1}^{n} \sum_{j=0}^{i-1} v_i(y^j \otimes y^{i-j}) - \sum_{i=1}^{n} \sum_{j=1}^{i} v_i(y^j \otimes y^{i-j}) \\
&= \sum_{i=1}^{n} v_i \left[\sum_{j=0}^{i-1} y^j \otimes y^{i-j} - \sum_{j=1}^{i} y^j \otimes y^{i-j} \right] \\
&= \sum_{i=1}^{n} v_i(1 \otimes y^i - y^i \otimes 1) \\
&= \sum_{i=1}^{n} 1 \otimes v_i y^i - \sum_{i=1}^{n} v_i y^i \otimes 1 \\
&= 1 \otimes (-v_0) - (-v_0) \otimes 1 \\
&= 0.
\end{aligned}$$

Since $1 \otimes y^m - y^m \otimes 1 = (1 \otimes y - y \otimes 1)(1 \otimes y^{m-1} + y \otimes y^{m-2} + \cdots + y^{m-1} \otimes 1)$, we see that $\alpha(1 \otimes y^m - y^m \otimes 1) = 0$ for all $m \geq 1$. As pointed out before, R is generated as a V-module by the powers of y, so this shows that $\alpha(1 \otimes r - r \otimes 1) = 0$ for every element r in R. Since J is generated over $R \otimes_V R$ by elements of the form $1 \otimes r - r \otimes 1$, this shows that $\alpha J = 0$. □

Our interest in the Noether different $N_V(R)$ stems from the fact that reduction modulo a nonzerodivisor x contained in $N_V(R)$ induces an embedding of the category of MCM R-modules into the category of $R/(x)$-modules. Such an element x is called an *efficient parameter* by Yoshino [41]. The embedding will preserve indecomposability and multiplicity, and will allow us to apply a lemma due to Harada–Sai [18] to prove a version of the Brauer–Thrall theorem.

The key fact about the Noether different is the following from [29].

Lemma 3.1.3. *Let V and R be as above, and let M be an $R \otimes_V R$-module. Then $N_V(R)$ annihilates the Hochschild cohomology $H^i_V(R, M)$ for all $i > 0$.*

Proposition 3.1.4. *Let V and R be as above, and let M and N be MCM R-modules. Assume that $\pi^t \in N_V(R)$ for some $t \geq 1$. Then for any homomorphism of $R/(\pi^{2t})$-*

modules $\varphi : M/\pi^{2t}M \longrightarrow N/\pi^{2t}N$*, there exists a homomorphism* $\psi : M \longrightarrow N$ *such that* $\varphi \otimes_R R/(\pi^t) = \psi \otimes_R R/(\pi^t)$*.*

Proof. This proof is based on [41, 6.15]. Note that since π is a nonzerodivisor in R and N is a MCM R-module, π is a nonzerodivisor on N. We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\pi^{2t}} & N & \longrightarrow & N/\pi^{2t}N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^t} & & \| & & \downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow[\pi^t]{} & N & \longrightarrow & N/\pi^t N & \longrightarrow & 0 \end{array} \tag{3.1.2}$$

By the Auslander–Buchsbaum formula, both M and N are finitely generated free V-modules. Applying $\mathrm{Hom}_V(M, -)$ gives the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_V(M,N) & \xrightarrow{\pi^{2t}} & \mathrm{Hom}_V(M,N) & \longrightarrow & \mathrm{Hom}_V(M,N/\pi^{2t}N) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi^t} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_V(M,N) & \xrightarrow[\pi^t]{} & \mathrm{Hom}_V(M,N) & \longrightarrow & \mathrm{Hom}_V(M,N/\pi^t N) & \longrightarrow & 0 \end{array} \tag{3.1.3}$$

Note that for any R-modules A and B, $\mathrm{Hom}_V(A, B)$ is an $R \otimes_V R$-module with the left structure induced from the R-action on B and the right structure induced from that on A. We take Hochschild cohomology $H^*_V(R, -)$. By [29], $H^0_V(R, \mathrm{Hom}_V(A, B)) \cong \mathrm{Hom}_R(A, B)$.

$$\begin{array}{ccccc} \mathrm{Hom}_R(M,N) & \longrightarrow & \mathrm{Hom}_R(M,N/\pi^{2t}N) & \longrightarrow & H^1_V(R,\mathrm{Hom}_V(M,N)) \\ \| & & \downarrow & & \downarrow{\scriptstyle \pi^t} \\ \mathrm{Hom}_R(M,N) & \longrightarrow & \mathrm{Hom}_R(M,N/\pi^t N) & \longrightarrow & H^1_V(R,\mathrm{Hom}_V(M,N)) \end{array} \tag{3.1.4}$$

Since $\pi^t \in N_V(R)$, π^t kills the Hochschild cohomology $H^1_V(R, \mathrm{Hom}_V(M, N))$. An easy diagram chase then shows that for any $\varphi \in \mathrm{Hom}_R(M, N/\pi^{2t}N)$, there exists $\psi \in \mathrm{Hom}_R(M, N)$ such that φ and ψ agree modulo π^t. □

Corollary 3.1.5. *Let* V *and* R *be as above, and assume that* $\pi^t \in N_V(R)$ *for some* $t \geq 1$*. Let* M *be a MCM* R*-module. Then* M *is indecomposable if and only if* $M/\pi^{2t}M$ *is indecomposable.*

Proof. See [41, 6.16]. □

• For the rest of this section, assume that there exists an integer t such that $\pi^t \in N_V(R)$. Further assume that R is an isolated singularity, and let Γ be the AR quiver for R. Let Γ^0 be a connected component, and assume that Γ^0 has bounded multiplicity type, that is, there exists an integer a such that $e(M) \leq a$ for any vertex $[M]$ in Γ^0. Then for any such M, the length $\ell(M/\pi^{2t}M)$ is bounded by ab, where b is the smallest integer such that $(\pi, y)^b \subseteq \pi^{2t}R$ [41, 1.7].

In what follows, call a homomorphism φ between two R-modules *trivial modulo* π^{2t} if $\varphi \otimes_R R/(\pi^{2t}) = 0$. The next result is referred to as a Harada–Sai Lemma in [40]. The original Harada–Sai Lemma is as follows [18]: Let S be an Artinian ring and let N_i, $0 \leq i \leq 2^r$, be indecomposable nonzero finitely generated S-modules such that $\ell(N_i) \leq r$ for $i = 0, \ldots, 2^r$. Let $g_i : N_{i-1} \longrightarrow N_i$, $i = 1, \ldots, 2^r$, be homomorphisms which are not isomorphisms. Then the composition $g_{2^r} g_{2^r-1} \cdots g_2 g_1$ is zero.

Lemma 3.1.6. *[40, 6.20] Keep the notation introduced thus far. Let M_i, $0 \leq i \leq 2^r$, be indecomposable MCM R-modules, and let $f_i : M_{i-1} \longrightarrow M_i$, $i = 1, \ldots, 2^r$, be homomorphisms which are not isomorphisms. Assume that $\ell(M_i/\pi^{2t}M_i) \leq r$ for $i = 0, \ldots, 2^r$. Then the composition $f_{2^r} f_{2^r-1} \cdots f_2 f_1$ is trivial modulo π^{2t}.*

Proof. In order to apply the original Harada–Sai Lemma to $S = R/(\pi^{2t})$, $N_i = M_i/\pi^{2t}M_i$, and $g_i = f_i \otimes_R S$, we need only show that $M_i/\pi^{2t}M_i$ is indecomposable for $i = 0, \ldots, 2^r$, and that no $f_i \otimes_R S$ is an isomorphism. The first statement follows from Corollary 3.1.5. For the second, assume that $f_i \otimes_R S$ is an isomorphism for some i. Then by [8, 21.13], f_i is an isomorphism, a contradiction. □

Lemma 3.1.7. *Keep the notation introduced thus far. Let M and N be two indecomposable MCM R-modules, and let $\varphi : M \longrightarrow N$ be a homomorphism that is not trivial modulo π^{2t}. Then $[M] \in \Gamma^0$ iff $[N] \in \Gamma^0$. Moreover, if either is in Γ^0, then $[M]$ and $[N]$ are connected by a path in Γ^0 of length less than 2^{ab}.*

Proof. First assume that $[N]$ is in Γ^0. Fix a non-negative integer n, to be determined later. Assume there is no path Π in the *undirected* graph Γ such that (1) Π connects $[M]$ to $[N]$ and (2) Π has length strictly less than n. We claim that there is a chain of homomorphisms between indecomposable MCM R-modules

$$M \xrightarrow{g} N_n \xrightarrow{f_n} N_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N \tag{3.1.5}$$

such that each f_i is irreducible and the composition $f_1 f_2 \cdots f_n g$ is not trivial modulo π^{2t}.

We construct the chain (3.1.5) by induction on n. If $n = 0$, then we take $g = \varphi$, so there is nothing to show. Assume $n \geq 1$. By the induction hypothesis, there is a chain

$$M \xrightarrow{g} N_{n-1} \xrightarrow{f_{n-1}} N_{n-2} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N \tag{3.1.6}$$

such that each f_i, $i = 1, \cdots, n-1$ is irreducible, each N_i is indecomposable, and the composition $f_1 f_2 \cdots f_{n-1} g$ is not trivial modulo π^{2t}. The assumption that there is no chain in the AR quiver of length less than n implies that g is not an isomorphism. We will extend this chain.

First suppose that N_{n-1} is not free. Then there is an AR sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{q} N_{n-1} \longrightarrow 0 \tag{3.1.7}$$

ending in N_{n-1}. Write E as a direct sum of indecomposable MCM R-modules, $E = \bigoplus_{i=1}^{s} E_i$. Then we can decompose $q = \sum_{i=1}^{s} q_i$, where each $q_i : E_i \longrightarrow N_{n-1}$ is an irreducible homomorphism by Lemma 3.0.2. The homomorphism $g : M \longrightarrow N_{n-1}$ is not an isomorphism, so is not a split injection since both modules are indecomposable. The defining property of the AR sequence ending in N_{n-1} then implies the existence of a homomorphism $h : M \longrightarrow E$ such that the triangle

$$\begin{array}{ccc} E & \xrightarrow{\quad q \quad} & N_{n-1} \\ & {}_{h}\nwarrow \quad \nearrow_{g} & \\ & M & \end{array}$$

commutes. Write $h = \sum_{i=1}^{s} h_i$ for homomorphisms $h_i : M \longrightarrow E_i$. Since (3.1.7) is not split, no h_i is an isomorphism. Since $f_1 f_2 \cdots f_{n-1} g$ is not trivial modulo π^{2t}, $f_1 f_2 \cdots f_{n-1}(qh)$ is not trivial modulo π^{2t}. Then $f_1 f_2 \cdots f_{n-1}(q_j h_j)$ is not trivial modulo π^{2t} for some j, $1 \leq j \leq s$, and we have the chain of homomorphisms

$$M \xrightarrow{h_j} E_j \xrightarrow{q_j} N_{n-1} \xrightarrow{f_{n-1}} N_{n-2} \cdots \longrightarrow N_1 \xrightarrow{f_1} N_0 = N$$

such that p_j is an irreducible homomorphism between indecomposable MCM modules, h_j is not an isomorphism, and the composition is not trivial modulo π^{2t}. We have extended the chain and completed the proof of the claim in the case where N_{n-1} is not free.

Now suppose that $N_{n-1} \cong R$ is free. Then since g is not an isomorphism, $g(M) \subseteq$

$\mathfrak{m}$, the maximal ideal of R. Since $\dim(R) = 1$, $\mathfrak{m}$ is a MCM R-module, and we have

$$\begin{array}{ccc} M & \xrightarrow{\quad g \quad} & R \\ & {\scriptstyle g'} \searrow \quad \nearrow {\scriptstyle h} & \\ & \mathfrak{m} & \end{array}$$

where h is the natural inclusion. We claim that h is irreducible. Suppose there is a factorization

$$\begin{array}{ccc} \mathfrak{m} & \xrightarrow{\quad h \quad} & R \\ & {\scriptstyle \alpha} \searrow \quad \nearrow {\scriptstyle \beta} & \\ & X & \end{array}$$

with X a MCM R-module. If β is not a split surjection, then $\beta(x) = x$ for each $x \in \mathfrak{m}$, and $\beta\alpha(\mathfrak{m}) = \mathfrak{m}$, so α is a split monomorphism. This shows that the inclusion $h : \mathfrak{m} \longrightarrow R$ is irreducible. Decompose $\mathfrak{m} = \bigoplus_{i=1}^{s} E_i$ with each E_i indecomposable, and write $g' = \sum_{i=1}^{s} g'_i$, $h = \sum_{i=1}^{s} h_i$ for maps $g'_i : M \longrightarrow E_i$ and $h_i : E_i \longrightarrow R$. Then, as before, $f_1 f_2 \cdots f_{n-1} h_j g'_j$ is nontrivial modulo π^{2t} for some j, and each h_j is irreducible. This extends the chain (3.1.6) and completes the proof of the claim.

Suppose now that $[M] \notin \Gamma^0$. Put $n = 2^{ab}$. Since there is no path Π in Γ of length less than n that connects $[N]$ to $[M]$, we obtain the chain of homomorphisms (3.1.5). Since $f_1 f_2 \cdots f_n g$ is non-trivial mod π^{2t}, so is $f_1 f_2 \cdots f_n$, and we have a contradiction to Lemma 3.1.6.

Suppose, conversely, that $[M] \in \Gamma^0$. We use an argument exactly dual to the one above to prove that $[N] \in \Gamma^0$: The claim this time is that if there is no path of length less than n connecting $[M]$ to $[N]$, then there is chain of homomorphisms between indecomposable MCM R-modules

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{g} N \tag{3.1.8}$$

such that each f_i is an irreducible homomorphism, g is not an isomorphism, and the composition is not trivial modulo π^{2t}. □

Theorem 3.1.8. *Let (V, π) be a complete discrete valuation ring with algebraically closed residue field, and set $R = V[[y]]/(f)$ for some non-zero non-unit $f \in V[[y]]$. Assume that R is an isolated singularity, that π is not a factor of f, and that $\pi^t \in N_V(R)$ for some t. Let Γ be the Auslander–Reiten quiver of R, and let Γ^0 be a nonempty connected component of Γ with bounded multiplicity type. Then $\Gamma^0 = \Gamma$*

and Γ *is a finite graph. In particular,* R *has finite CM type.*

Proof. Let $M \in \Gamma^0$. Then, by Nakayama's lemma, there exists $x \in M \setminus \pi^{2t}M$, so there is a homomorphism $R \longrightarrow M$, taking 1 to x, such that f is nontrivial modulo π^{2t}. Lemma 3.1.7 shows that $[R] \in \Gamma^0$. Then for any $[N] \in \Gamma$ we can define a homomorphism $R \longrightarrow N$ in the same way and deduce that $[N] \in \Gamma^0$.

To see that Γ is a finite graph, note that by Lemma 3.1.7, any vertex of Γ is connected to $[R]$ by a chain of arrows of length less than 2^{ab}. Since Γ is a locally finite graph (Lemma 3.0.5), Γ is finite. □

3.2 Mixed ADE Singularities

The goal of this section is to compute the Auslander–Reiten quivers of the mixed ADE singularities, and thereby show that they have finite CM type. The mixed ADE singularities are the natural generalizations of the simple plane curve singularities over a field, which are known to be precisely those plane curve singularities of finite CM type (see Chapter 0). All our proofs in this section are modeled on those in [40] and [41].

Definitions and Preliminaries

Throughout this section, we keep the notation of Section 3.1: Let (V, π) be a complete discrete valuation ring of characteristic zero and residual characteristic $p > 0$. Let $R = V[[y]]/(f)$ be a hypersurface over V, where f is a non-zero non-unit of $S = V[[y]]$. We always assume that f is square-free, that is, R is an isolated singularity.

Definition 3.2.1. *We say that* R *is a* mixed ADE singularity *if* R *is isomorphic to one of the following. These rings are defined only with the listed restrictions on the residue field characteristic* p*. Also included are the derivatives of the defining equations with respect to* y*.*

Name	f		f'	restriction
(A_n)	$y^2+\pi^{n+1}$	$(n \geq 2)$	$2y$	$p \neq 2$
(A'_n)	π^2+y^{n+1}	$(n \geq 2)$	$(n+1)y^n$	$p \not| n+1$
(D_n)	$\pi(y^2+\pi^{n-2})$	$(n \geq 4)$	$2\pi y$	$p \neq 2$
(D'_n)	$y(\pi^2+y^{n-2})$	$(n \geq 4)$	$\pi^2+(n-1)y^{n-2}$	$p \not| n-2$
(E_6)	$y^3+\pi^4$		$3y^2$	$p \neq 3$
(E'_6)	π^3+y^4		$4y^3$	$p \neq 2$
(E_7)	$y(y^2+\pi^3)$		$3y^2+\pi^3$	$p \neq 2$
(E'_7)	$\pi(\pi^2+y^3)$		$3\pi y^2$	$p \neq 3$
(E_8)	$y^3+\pi^5$		$3y^2$	$p \neq 3$
(E'_8)	π^3+y^5		$5y^4$	$p \neq 5$

In order to apply Theorem 3.1.8 to conclude that the Auslander–Reiten quivers of these rings are connected, we need to know that $\pi^t \in N_V(R)$ for some integer t.

Lemma 3.2.2. *If R is a mixed ADE singularity, but not (D_n) or ($E_7{}'$), then some power of π is a nonzerodivisor contained in $N_V(R)$.*

The reason for excluding the two cases (D_n) and (E'_7) is that π is a zerodivisor in those rings.

Proof. For most cases, the statement is clear from the table; use the fact that the derivative f' is in $N_V(R)$ (Lemma 3.1.2). The exceptions are the cases (D'_n) and (E_7). The derivatives in those cases are unpleasant enough that it is not immediately obvious that a power of π is in the ideal generated by the derivative. We explicitly write out equations to take care of these cases.

For (D'_n), note that

$$yf' = \pi^2 y + (n-1)y^{n-1} = (n-2)y^{n-1},$$

so if $p \not| \; n-2$, then $y^{n-1} \in N_V(R)$. Also,

$$\pi^{2(n-1)}f' = \pi^{2n} + (n-1)\pi^{2(n-1)}y^{n-1}$$

so $\pi^{2n} \in N_V(R)$. Now, for (E_7), we have

$$yf' = 3y^3 + \pi^3 y = 2y^3,$$

so that if $p \neq 2$, $y^3 \in N_V(R)$. Also

$$\pi^6 f' = 3\pi^6 y^2 + \pi^9 = -3y^6 + \pi^9$$

so π^9 is in $N_V(R)$. This finishes the proof of Lemma 3.2.2. □

We now briefly review some relevant facts about Auslander–Reiten quivers in the specific context of this section.

Lemma 3.2.3. *The AR translation of a nonfree indecomposable MCM R-module M is given by* $\tau(M) \cong \operatorname{syz}^1_R(M)$.

Proof. In general (Lemma 3.0.4), $\tau(M) \cong (\operatorname{syz}^d_R \operatorname{tr}(M))'$, where $d = \dim(R)$, tr is the Auslander transpose of M, and $(-)'$ means the canonical dual $\operatorname{Hom}_R(-, \omega_R)$. By definition of the Auslander translation, dualizing a minimal free presentation $F_1 \to F_0 \to M \to 0$ of M into R gives

$$0 \longrightarrow M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \operatorname{tr}(M) \longrightarrow 0. \tag{3.2.1}$$

Since the original resolution was minimal, (3.2.1) is as well and therefore is a minimal free resolution of $\operatorname{tr}(M)$. The first syzygy in this sequence is thus $\operatorname{syz}^1_R(\operatorname{tr}(M))$, which, since R is Gorenstein, is isomorphic to $\tau(M)^*$. Dualizing again gives $0 \to \tau(M) \to F_0 \to M \to 0$, so $\tau(M) = \operatorname{syz}^1_R(M)$. □

We also use the following two more general facts. The first tells us how to identify the AR sequence ending in a given module if we run into it on the road, and the second gives a map to find it.

Let M be a nonfree indecomposable MCM R-module. The AR sequence ending in M can be represented by an element of $\operatorname{Ext}^1_R(M, \tau(M))$. Since M is locally free on the punctured spectrum of R (Lemma 3.0.1), $\operatorname{Ext}^1_R(M, \tau(M))$ has finite length. In fact, the proof of [41, 3.11] shows that the socle of $\operatorname{Ext}^1_R(M, \tau(M))$ is a one-dimensional vector space over the residue field of R. Choose a generator s for this socle. Then [41, 3.11] s represents the AR sequence ending in M.

The second fact deals with the theory of matrix factorizations over the hypersurface $R = S/(f)$. We refer the reader to [41, Chapter 7] for the details.

Let M be a MCM R-module with no nonzero free summands. There is an exact sequence of S-modules

$$0 \longrightarrow S^m \xrightarrow{\varphi} S^m \longrightarrow M \longrightarrow 0$$

where m is the number of generators required for M. We can regard φ as an $m \times m$ matrix with entries in the maximal ideal of S. There is another $m \times m$ matrix ψ such that both compositions $\varphi\psi$ and $\psi\varphi$ are equal to f times the identity matrix. The pair (φ, ψ) is called the *reduced matrix factorization* corresponding to M, and we write $M = \operatorname{coker}(\varphi, \psi)$.

Suppose now that N is another MCM R-module with no nonzero free summand, with corresponding reduced matrix factorization (φ', ψ'), and suppose $h : N \to \operatorname{syz}^1_R(M)$ is a homomorphism. Since the resolution of M is periodic of period 2 [41, Chapter 7], $\operatorname{syz}^1_R(M) = \operatorname{coker}(\psi, \varphi)$. We can choose homomorphisms α and β to make the following diagram commute:

$$\begin{array}{ccccccc} S^m & \xrightarrow{\varphi} & S^m & \longrightarrow & N & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle h} & & \\ S^n & \xrightarrow[\psi]{} & S^n & \longrightarrow & \operatorname{syz}^1_R(M) & \longrightarrow & 0. \end{array} \tag{3.2.2}$$

Since M is its own second syzygy, we have an exact sequence

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow \operatorname{syz}{}^1_R(M) \longrightarrow 0. \tag{3.2.3}$$

Applying $\operatorname{Hom}_R(N, -)$ induces a surjection $\rho : \operatorname{Hom}_R(N, \operatorname{syz}^1_R(M)) \to \operatorname{Ext}^1_R(N, M)$ (recall that R is a hypersurface, hence Gorenstein, so $\operatorname{Ext}^1_R(N, R^n) = 0$). Now, the image of the map h under ρ can be represented by a short exact sequence $0 \to M \to L \to N \to 0$, which corresponds to a pullback of (3.2.3) by h.

Then L is a MCM R-module, and [41, 7.8] the reduced matrix factorization corresponding to L is $\left(\begin{bmatrix} \varphi & \beta \\ 0 & \varphi' \end{bmatrix}, \begin{bmatrix} \psi & -\alpha \\ 0 & \psi' \end{bmatrix} \right)$.

The (A_n) singularities, n even

Let $R = V[[y]]/(y^2 + \pi^{n+1})$, where $n \geq 2$ is an *even* integer. Assume that the residue field characteristic p is not equal to 2. Then R is a singularity of type (A_n). We will show that R has finite CM type. The polynomial $y^2 + \pi^{n+1}$ has no linear factors, since $n+1$ is odd. Therefore, we have matrix factorizations of $y^2+\pi^{n+1}$ of the form

$$\varphi_j = \begin{bmatrix} y & \pi^j \\ \pi^{n-j+1} & -y \end{bmatrix}, \qquad 0 \le j \le n+1, \tag{3.2.4}$$

and we will see that these are all the matrix factorizations up to equivalence. Set $M_j = \operatorname{coker} \varphi_j$. Since elementary row and column operations transform φ_j into φ_{n-j}, $M_j \cong M_{n-j+1}$ for $0 \le j \le n/2$. Further, each M_j is indecomposable; a decomposition would lead to a linear factorization of f. Finally, note that $M_0 \cong R$, and each M_j is isomorphic to the ideal $(y, \pi^j)R$.

Let's compute the AR sequence ending in M_j. Choose $j \ge 1$. Since M_j is its own first syzygy, we have an exact sequence

$$0 \longrightarrow M_j \longrightarrow R^2 \longrightarrow M_j \longrightarrow 0. \tag{3.2.5}$$

Recall that M_j is isomorphic to the ideal (y, π^j). Consider the two endomorphisms of M_j given by multiplication by $-y$ and multiplication by π^n. I claim that the pullback of (3.2.5) given by either of these endomorphisms is a split exact sequence. We need only show that each of these endomorphisms of M_j factors through the free module R^2. Define a map $M_j \to R^2$ by $x \mapsto \left(\begin{smallmatrix} -x \\ 0 \end{smallmatrix}\right)$; then composition with the surjection $\left[\begin{smallmatrix} y & \pi^j \end{smallmatrix}\right] : R^2 \to M_j$ is equal to multiplication by $-y$. On the other hand, the map $M_j \to R^2$ taking $x \in M_j$ to $\left(\begin{smallmatrix} 0 \\ \pi^{n-j}x \end{smallmatrix}\right)$ gives a factorization of the map given by multiplication by π^n through the free module R^2.

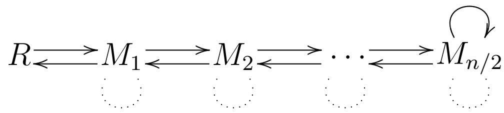
Let h be the endomorphism of M_j defined by multiplication by π^n/y, an element of the total quotient ring of R. Then yh is multiplication by π^n, and πh is multiplication by $-y$. By the previous paragraph, the images of yh and πh in $\operatorname{Ext}^1_R(M_j, M_j)$ are zero. This shows that the image of h is in the socle of $\operatorname{Ext}^1_R(M_j, M_j)$. If we show that the pullback of (3.2.5) by h is not a split sequence, then we will have shown that the image of h in $\operatorname{Ext}^1_R(M_j, M_j)$ generates the socle. We can take $\beta = -\alpha = \left[\begin{smallmatrix} 0 & \pi^{j-1} \\ -\pi^{n-j} & 0 \end{smallmatrix}\right]$ in 3.2.2 to represent h as a pair of maps between free modules. Thus pulling back by h gives a short exact sequence $0 \to M_j \to L \to M_j \to 0$, where

$$L = \operatorname{coker} \begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix}.$$

It is a fairly straightforward matrix-equivalence computation to check that L is isomorphic to $M_{j-1} \oplus M_{j+1}$. To wit:

$$\begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix} \sim \begin{bmatrix} y & 0 & 0 & \pi^{j-1} \\ 0 & -y & -\pi^{n-j} & 0 \\ y\pi & -\pi^j & y & \pi^j \\ \pi^{n-j+2} & y\pi & \pi^{n-j+1} & -y \end{bmatrix} \sim$$
$$\begin{bmatrix} y & 0 & 0 & \pi^{j-1} \\ 0 & -y & -\pi^{n-j} & 0 \\ 0 & -\pi^j & y & 0 \\ \pi^{n-j+2} & 0 & 0 & -y \end{bmatrix} \sim \begin{bmatrix} y & \pi^{j-1} & 0 & 0 \\ \pi^{n-j+2} & -y & 0 & 0 \\ 0 & 0 & -y & -\pi^{n-j} \\ 0 & 0 & -\pi^j & y \end{bmatrix}$$

Since the result of the pullback by h is not split, the AR sequence ending in M_j is indeed $0 \to M_j \to M_{j-1} \oplus M_{j+1} \to M_j \to 0$. Hence we can draw a connected component of the AR quiver for R.



(A_n) for even n

By Theorem 3.1.8, this is the complete quiver. Thus R has finite CM type.

The ($\mathbf{A}_n$) singularities, n odd

Let $R = V[[y]]/(y^2 + \pi^{n+1})$, with n an *odd* positive integer. Assume that V has residue field characteristic greater than 2. Then V contains an element i such that $i^2 = -1$ (the residue field does, and use Hensel's Lemma to lift it back up to V). Now, R is no longer a domain, for we have $y^2 + \pi^{n+1} = (\pi^{(n+1)/2} + iy)(\pi^{(n+1)/2} - iy)$.

Set

$$\begin{aligned} N_+ &= R/(\pi^{(n+1)/2} + iy) \\ N_- &= R/(\pi^{(n+1)/2} - iy) \\ \varphi_j &= \begin{bmatrix} y & \pi^j \\ \pi^{n-j+1} & -y \end{bmatrix}, \quad 1 \le j \le n+1 \\ M_j &= \operatorname{coker} \varphi_j. \end{aligned}$$

Then, as before, $M_j \cong M_{n-j+1}$ is an ideal for $j = 1, \ldots, n+1$, and $M_0 \cong R$. Furthermore,

$$\varphi_{(n+1)/2} = \begin{bmatrix} y & \pi^{(n+1)/2} \\ \pi^{(n+1)/2} & -y \end{bmatrix} \sim$$

$$\begin{bmatrix} y - i\pi^{(n+1)/2} & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} & -y \end{bmatrix} \sim \begin{bmatrix} 0 & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} - iy & -y \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi^{(n+1)/2} + iy \\ \pi^{(n+1)/2} - iy & -2y \end{bmatrix} \sim \begin{bmatrix} 0 & \pi^{(n+1)/2} - iy \\ \pi^{(n+1)/2} - iy & -y + i\pi^{(n+1)/2} \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi^{(n+1)/2} - iy \\ \pi^{(n+1)/2} - iy & 0 \end{bmatrix}$$

so $M_{(n+1)/2} \cong N_+ \oplus N_-$.

Since they arise as matrix factorizations, N_+, N_-, and M_j are all MCM R-modules. The AR translations are given by $\tau(-) = \mathrm{syz}^1_R(-)$, so $\tau(M_j) \cong M_j$, $\tau(N_+) \cong N_-$, and $\tau(N_-) \cong N_+$.

As in the case where n is even, the AR sequence for M_j is $0 \to M_j \to L \to M_j \to 0$ where

$$L = \operatorname{coker} \begin{bmatrix} y & \pi^j & 0 & \pi^{j-1} \\ \pi^{n-j+1} & -y & -\pi^{n-j+1} & 0 \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j+1} & -y \end{bmatrix} \cong M_{j-1} \oplus M_{j+1}.$$

To compute the AR sequence ending in N_+, consider the endomorphism h of N_+ given by multiplication by $\pi^{(n-1)/2}$. We have $\pi h = \pi^{(n+1)/2}$ and $yh = y\pi^{(n-1)/2}$. Pulling back the short exact sequence $0 \to N_- \to R \to N_+ \to 0$ via $2iy$ gives a middle term with presentation matrix

$$\begin{bmatrix} \pi^{(n+1)/2} + iy & 2iy \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix} \sim \begin{bmatrix} \pi^{(n+1)/2} + iy & 0 \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix},$$

so multiplication by $2iy$ splits the resolution of N_+. It follows that y splits the sequence, since $2i$ is a unit, and so $y\pi^{n/2-1}$ does as well. We also have $yN_+ = y(\pi^{(n+1)/2} - iy) = \pi^{(n+1)/2}(\pi^{(n+1)/2} - iy)$, so $yN_+ = \pi^{(n+1)/2}N_+$. This shows that the map $\pi h = \pi^{(n+1)/2}$ also splits the exact sequence $0 \to N_- \to R \to N_+ \to 0$. It remains only to show that pulling back via h does not split the sequence, and we will have that h generates the socle of $\operatorname{Ext}^1_R(N_+, N_-)$. The sequence obtained by pulling back via h is $0 \to N_- \to P \to N_+ \to 0$, where

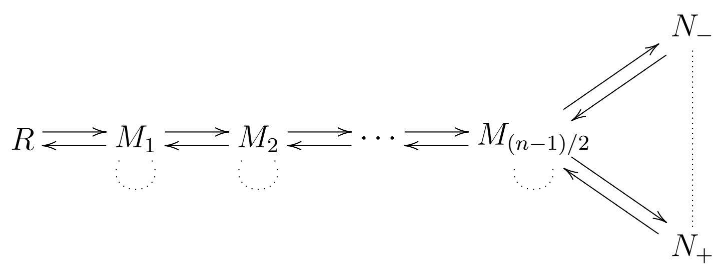
$$P = \operatorname{coker}\begin{bmatrix} \pi^{(n+1)/2} + iy & \pi^{(n-1)/2} \\ 0 & \pi^{(n+1)/2} - iy \end{bmatrix} \cong$$

$$\operatorname{coker}\begin{bmatrix} iy & \pi^{(n-1)/2} \\ -\pi^{(n-1)/2} + iy\pi & \pi^{(n+1)/2} - iy \end{bmatrix} \cong \operatorname{coker}\begin{bmatrix} iy & \pi^{(n-1)/2-1} \\ -\pi^{(n-1)/2} & -iy \end{bmatrix} \cong$$

$$\cong M_{n/2-1}.$$

Since this is nonsplit, $0 \to N_- \to M_{(n-1)/2} \to N_+ \to 0$ is the AR sequence ending in N_+. Taking syzygies, we see that the AR sequence ending in N_- is $0 \to N_+ \to M_{(n-1)/2} \to N_- \to 0$.

Thus a connected component of the AR quiver for R looks like



(A_n) for odd n

By Theorem 3.1.8, this is the whole quiver, and so R has finite CM type. This completes the (A_n) singularities.

The ($\mathbf{A}'_n$) singularities

Let $R = V[[y]]/(\pi^2 + y^{n+1})$. Assume that the residue field characteristic p does not divide $n+1$. The matrix calculations of the previous section hold true if y and π are interchanged, so R has finite CM type.

The (D_n) singularities, n odd

Let $R = V[[y]]/(y^2\pi + \pi^{n-1})$, where $n \geq 4$ is an odd integer. Assume that the residue field characteristic p is not equal to 2. Set

$$\begin{aligned}
\alpha &= [\pi] \\
\beta &= [y^2 + \pi^{n-2}] \\
\varphi_j &= \begin{bmatrix} y & \pi^j \\ \pi^{n-j-2} & -y \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\psi_j &= \begin{bmatrix} y\pi & \pi^{j+1} \\ \pi^{n-j-1} & -y\pi \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\chi_j &= \begin{bmatrix} y & \pi^j \\ \pi^{n-j-1} & -y\pi \end{bmatrix}, \quad 0 \leq j \leq n-3 \\
\eta_j &= \begin{bmatrix} y\pi & \pi^j \\ \pi^{n-j-1} & -y \end{bmatrix}, \quad 0 \leq j \leq n-3.
\end{aligned} \tag{3.2.6}$$

It is easy to check that $(\alpha, \beta), (\beta, \alpha), (\varphi_j, \psi_j), (\psi_j, \varphi_j), (\chi_j, \eta_j), (\eta_j, \chi_j)$ are all matrix factorizations of $y^2\pi + \pi^{n-1}$. Put

$$\begin{aligned}
A &= \operatorname{coker} \alpha, \quad B = \operatorname{coker} \beta \\
M_j &= \operatorname{coker} \varphi_j, \quad N_j = \operatorname{coker} \psi_j \\
X_j &= \operatorname{coker} \chi_j, \quad Y_j = \operatorname{coker} \eta_j.
\end{aligned} \tag{3.2.7}$$

There is some collapsing here: $M_0 \cong B \oplus R$, $N_0 \cong A$, and $X_0 \cong Y_0 \cong R$. Also, $X_{(n-1)/2} \cong Y_{(n-1)/2}$, $M_j \cong M_{n-j-2}$, $N_j \cong N_{n-j-2}$, $X_j \cong Y_{n-j-1}$, and $Y_j \cong X_{n-j-1}$. Finally, before we dive into the AR sequences, note that M_j is isomorphic to the ideal $(y\pi, \pi^{j+1})R$, and Y_j is isomorphic to $(y, \pi^j)R$.

Using the fact that $\tau(-) \cong \operatorname{syz}^1_R(-)$ by Lemma 3.0.4, we can see that our collection of modules is closed under AR translations. Let's compute the AR sequences.

Note that B is isomorphic to the ideal $(\pi)R$. Consider the first part of a free resolution of B: $0 \to A \to R \to B \to 0$. The endomorphism of B given by multiplication by y^2 factors through the free module R via $x \mapsto (y^2x/\pi)$. (Note that the division here is legal since $x \in B$.) Similarly, the map on B given by $y\pi$ factors through R via $x \mapsto yx$. Hence both of these endomorphisms of B split the resolution of B. We will show that the short exact sequence given by pulling back the map given by multiplication by y represents the socle element of $\operatorname{Ext}^1_R(B, A)$. As before, the middle term of this sequence has presenting matrix $\begin{bmatrix} \beta & y \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} y^2+\pi^{n-2} & y \\ 0 & \pi \end{bmatrix} \sim \begin{bmatrix} \pi^{n-2} & y \\ -y\pi & \pi \end{bmatrix}$. This is the presenting matrix for X_1, so the result of pulling back via y is nonsplit, and y is a nonzero socle element of $\operatorname{Ext}^1_R(B, A)$. Thus the AR sequence ending in A is $0 \to A \to X_1 \to B \to 0$. Taking syzygies gives the AR sequence ending in B: $0 \to A \to Y_1 \to B \to 0$.

On to the M_j's. Recall that $M_j \cong (y\pi, \pi^{j+1})R$. We have the exact sequence $0 \to N_j \to R^2 \to M_j \to 0$. The endomorphism of M_j given by multiplication by $-y\pi$ factors through the free module R^2 via $x \mapsto \begin{pmatrix} -x \\ 0 \end{pmatrix}$, while the map given by multiplication by y^2 factors through R^2 via $x \mapsto \begin{pmatrix} xy/\pi \\ 0 \end{pmatrix}$ (note that if $x \in M_j$, then $x \in \pi R$). Thus multiplication by both $-y\pi$ and y^2 factor through R^2, and so pulling back by either of these splits a free resolution of M_j. If we show that pulling back by $-y$ does not split that resolution, we will have identified the element that generates the socle of $\operatorname{Ext}^1_R(M_j, N_j)$.

The map given by multiplication by $-y$ has a matrix factorization (α, β), where $\beta = -\alpha = \begin{bmatrix} 0 & \pi^j \\ -\pi^{n-j-2} & 0 \end{bmatrix}$ and so the middle term of the short exact sequence obtained by pulling back via $-y$ has presenting matrix

$$\begin{bmatrix} \varphi_j & \beta \\ 0 & \psi_j \end{bmatrix} \sim \begin{bmatrix} y & \pi^j & 0 & \pi^j \\ \pi^{n-j-2} & -y & -\pi^{n-j-2} & 0 \\ 0 & 0 & y\pi & \pi^{j+1} \\ 0 & 0 & \pi^{n-j-1} & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} y & \pi^j & 0 & \pi^j \\ \pi^{n-j-2} & -y & -\pi^{n-j-2} & 0 \\ -y\pi & -\pi^{j+1} & y\pi & 0 \\ 0 & 0 & \pi^{n-j-1} & -y\pi \end{bmatrix} \sim \begin{bmatrix} y & \pi^j & 0 & \pi^j \\ 0 & -y & -\pi^{n-j-2} & 0 \\ 0 & -\pi^{j+1} & y\pi & 0 \\ \pi^{n-j-1} & 0 & \pi^{n-j-1} & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} y & \pi^j & 0 & \pi^j \\ 0 & -y & -\pi^{n-j-2} & 0 \\ 0 & -\pi^{j+1} & y\pi & 0 \\ \pi^{n-j-1} & -y\pi & 0 & -y\pi \end{bmatrix} \sim \begin{bmatrix} y & 0 & 0 & \pi^j \\ 0 & -y & -\pi^{n-j-2} & 0 \\ 0 & -\pi^{j+1} & y\pi & 0 \\ \pi^{n-j-1} & 0 & 0 & -y\pi \end{bmatrix} \sim$$

$$\begin{bmatrix} y & \pi^{j+1} & 0 & 0 \\ \pi^{n-j-2} & -y\pi & 0 & 0 \\ 0 & 0 & y\pi & \pi^j \\ 0 & 0 & \pi^{n-j-1} & -y \end{bmatrix}$$

which is the presenting matrix for $X_{j+1} \oplus Y_j$. This shows that the image of $-y$ generates the socle of $\mathrm{Ext}^1_R(M_j, N_j)$, and the AR sequence ending in M_j is $0 \to N_j \to X_{j+1} \oplus Y_j \to M_j \to 0$. For N_j, we can just take syzygies in the AR sequence ending in M_j. This gives $0 \to M_j \to Y_{j+1} \oplus X_j \to N_j \to 0$ for the AR sequence ending in N_j.

Next we consider Y_j, which is isomorphic to the ideal $(y, \pi^j)R$. Pull back the resolution $0 \to X_j \to R^2 \to Y_j \to 0$ via the map given by multiplication by $y\pi$ on Y_j. Since $y\pi$ factors through R^2 via $x \mapsto \left(\begin{smallmatrix} x\pi \\ 0 \end{smallmatrix}\right)$, the result splits. Now pull back by the map given by multiplication by y^2. Again, $x \mapsto \left(\begin{smallmatrix} yx \\ 0 \end{smallmatrix}\right)$ is a map $Y_j \to R^2$ which factors y^2, so the result splits.

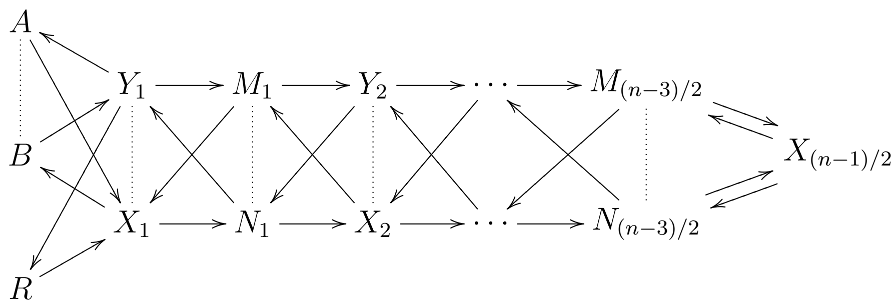
We now show that the result of pulling back by y is not split, so that multiplication by y on Y_j gives the socle element of $\mathrm{Ext}^1_R(Y_j, X_j)$, that is, the AR sequence ending in Y_j. We can write $y = \mathrm{coker}(\alpha, \beta)$, where $\alpha = \beta = \left(\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right)$. A presenting matrix for the middle term of the extension obtained by pulling back by y is thus

$$\begin{bmatrix} y\pi & \pi^j & y & 0 \\ \pi^{n-j-1} & -y & 0 & y \\ 0 & 0 & y & \pi^j \\ 0 & 0 & \pi^{n-j-1} & -y\pi \end{bmatrix} \sim \begin{bmatrix} y\pi & \pi^j & y & 0 \\ \pi^{n-j-1} & -y & 0 & y \\ -y & -\pi^j & 0 & \pi^j \\ \pi^{n-j} & -y\pi & \pi^{n-j-1} & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi^j & y & 0 \\ \pi^{n-j-1} & 0 & 0 & y \\ -y & 0 & 0 & \pi^j \\ 0 & -y\pi & \pi^{n-j-1} & 0 \end{bmatrix} \sim \begin{bmatrix} y & \pi^j & 0 & 0 \\ \pi^{n-j-1} & -y & 0 & 0 \\ 0 & 0 & -y\pi & \pi^{n-j-1} \\ 0 & 0 & \pi^j & -y \end{bmatrix}$$

This is the presentation matrix for $M_{j-1} \oplus N_j$, so the AR sequence ending in Y_j is $0 \to X_j \to M_{j-1} \oplus N_j \to Y_j \to 0$. As before, we take syzygies to see that the AR sequence ending in X_j is $0 \to Y_j \to N_{j-1} \oplus M_j \to X_j \to 0$.

This allows us to draw a connected component of the AR quiver for R. Unfortunately, we cannot use Theorem 3.1.8 to conclude that this is the entire quiver, since π is a zerodivisor in R.



(D_n) for odd n (incomplete)

The ($\mathbf{D}_n$) singularities, n even

Let $R = V[[y]]/(y^2\pi + \pi^{n-1})$, where $n \geq 4$ is an even integer. Assume the residue field characteristic p is not 2. Then, as in the A_n singularities with n even, V contains an element i whose square is -1. Define A, B, M_j, N_j, X_j, and Y_j as in (3.2.6) and (3.2.7). Also let

$$\begin{aligned} C_+ &= \operatorname{coker}(\pi(y + i\pi^{(n-2)/2})) \\ C_- &= \operatorname{coker}(\pi(y - i\pi^{(n-2)/2})) \\ D_= &= \operatorname{coker}(y - i\pi^{(n-2)/2}) \\ D_+ &= \operatorname{coker}(y + i\pi^{(n-2)/2}) \end{aligned} \tag{3.2.8}$$

Then, as in the case of n odd, $M_0 \cong B \oplus R$, $N_0 \cong A$, and $X_0 \cong Y_0 \cong R$. Also, $X_{(n-1)/2} \cong Y_{(n-1)/2}$, $M_j \cong M_{n-j-2}$, $N_j \cong N_{n-j-2}$, $X_j \cong Y_{n-j-1}$, and $Y_j \cong X_{n-j-1}$. Furthermore, M_j is isomorphic to the ideal $(y\pi, \pi^{j+1})R$, and Y_j is isomorphic to $(y, \pi^j)R$. In this case, however, $M_{(n-2)/2} \cong D_+ \oplus D_-$ and $N_{(n-2)/2} \cong C_+ \oplus C_-$.

We already know that we have AR sequences

$$\begin{aligned} 0 \to A \to &X_1 \to B \to 0 \\ 0 \to B \to &Y_1 \to A \to 0 \end{aligned} \tag{3.2.9}$$

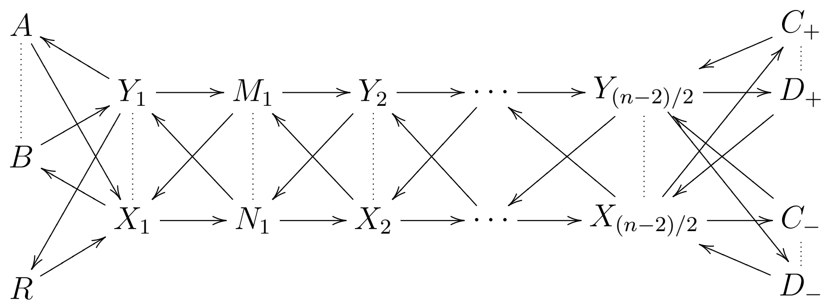
and, for $j \neq (n-2)/2$

$$\begin{aligned} 0 &\to N_j \to X_{j+1} \oplus Y_j \to M_j \to 0 \\ 0 &\to M_j \to Y_{j+1} \oplus X_j \to N_j \to 0 \\ 0 &\to X_j \to M_{j-1} \oplus N_j \to Y_j \to 0 \\ 0 &\to Y_j \to N_{j-1} \oplus M_j \to X_j \to 0 \end{aligned} \tag{3.2.10}$$

All that remains is to compute the AR sequences ending in $C_\pm$ and $D_\pm$. Clearly the AR translation of $C_\pm$ is $D_\pm$, and vice versa. From the decompositions $M_{(n-2)/2} \cong D_+ \oplus D_-$ and $N_{(n-2)/2} \cong C_+ \oplus C_-$ we get AR sequences

$$\begin{aligned} 0 \to D_+ &\to X_{(n-2)/2} \to C_+ \to 0 \\ 0 \to D_- &\to X_{(n-2)/2} \to C_- \to 0 \\ 0 \to C_+ &\to Y_{(n-2)/2} \to D_+ \to 0 \\ 0 \to C_- &\to Y_{(n-2)/2} \to D_- \to 0 \end{aligned}$$

Note that $Y_{(n-2)/2} \cong X_{n/2}$, so we get the following connected component of the AR quiver. As in the case where n is odd, however, we are unable to use Theorem 3.1.8, so cannot say that this is the whole quiver.



(D_n) for even n (incomplete)

The (D'_n) singularities

Let $R = V[[y]]/(y\pi^2 + y^{n-1})$ for an odd integer $n \geq 4$. Assume that the residue field characteristic p does not divide $n-2$. These are exactly symmetric to the (D_n) singularities; again, the matrix computations do not depend on π being a parameter of V. In this case, however, π is not a zerodivisor, so we can apply Theorem 3.1.8 to see that the singularities of type (D'_n) have finite CM type.

The ($\mathbf{E}_6$) singularity

Let $R = V[[y]]/(y^3 + \pi^4)$, assume $p \neq 3$, and let

$$\begin{aligned}
\varphi_1 &= \begin{bmatrix} y & \pi \\ \pi^3 & -y^2 \end{bmatrix} & \psi_1 &= \begin{bmatrix} y^2 & \pi \\ \pi^3 & -y \end{bmatrix} \\
\varphi_2 &= \begin{bmatrix} y & \pi^2 \\ \pi^2 & -y^2 \end{bmatrix} & \psi_2 &= \begin{bmatrix} y^2 & \pi^2 \\ \pi^2 & -y \end{bmatrix} \\
\alpha &= \begin{bmatrix} \pi^3 & y^2 & y\pi^2 \\ y\pi & -\pi^2 & y^2 \\ y^2 & -y\pi & -\pi^3 \end{bmatrix} & \beta &= \begin{bmatrix} \pi & 0 & y \\ y & -\pi^2 & 0 \\ 0 & y & -\pi \end{bmatrix} \\
\chi &= \begin{bmatrix} \varphi_2 & \begin{bmatrix} 0 & \pi \\ -y\pi & 0 \end{bmatrix} \\ 0 & \psi_2 \end{bmatrix} & \eta &= \begin{bmatrix} \psi_2 & \begin{bmatrix} 0 & y\pi \\ -\pi & 0 \end{bmatrix} \\ 0 & \varphi_2 \end{bmatrix}
\end{aligned}$$

Then each pair (φ_i, ψ_i), (α, β), (χ, η) is a matrix factorization of $y^3 + \pi^4$. Let $M_i = \operatorname{coker} \varphi_i$, $N_i = \operatorname{coker} \psi_i$, $A = \operatorname{coker} \alpha$, $B = \operatorname{coker} \beta$, $X = \operatorname{coker} \chi$, $Y = \operatorname{coker} \eta$.

We can identify these modules more clearly. It is easy to check that $M_1 \cong (y^2, \pi)R$, $N_1 \cong (y, \pi)R$, $N_2 \cong M_2 \cong (y^2, \pi^2)R$, $B \cong (y^2, y\pi, \pi^2)R$, and A has rank 2. Also, $X \cong Y$. Using the fact that $\tau(-) = \operatorname{syz}^1_R(-)$ we can see that our collection of modules is closed under AR translations. Let's compute the AR sequences.

Begin with $N_1 \cong (y, \pi)R$. A presentation of N_1 is given by $0 \to M_1 \to R^2 \to N_1 \to 0$. When we pull back along the endomorphism of N_1 given by multiplication by $-\pi^3$, we see that the map $N_j \to R^2$ given by $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi^2 x \end{smallmatrix}\right)$ factors $-\pi^3$ through a free module. Similarly, the map given by multiplication by y^2 on N_j factors through R^2 via $x \mapsto \left(\begin{smallmatrix} xy \\ 0 \end{smallmatrix}\right)$.

We will show that the endomorphism h of N_1 given by multiplication by $-\pi^3/y$ gives the socle element of $\operatorname{Ext}^1_R(N_1, M_1)$, that is, the AR sequence ending in N_1. Note that $yh = -\pi^3$ and $\pi h = y^2$, so we need only show that pulling back by h does not split the short exact sequence $0 \to M_1 \to R^2 \to N_1 \to 0$. We can write $h = \operatorname{coker}(\gamma, \delta)$, where $\gamma = \left[\begin{smallmatrix} 0 & 1 \\ -\pi^2 & 0 \end{smallmatrix}\right]$ and $\delta = \left[\begin{smallmatrix} 0 & -1 \\ y\pi^2 & 0 \end{smallmatrix}\right]$. A presenting matrix for the middle term of the AR sequence ending in N_1 is

$$\begin{bmatrix} \varphi_1 & \begin{bmatrix} 0 & -1 \\ y\pi^2 & 0 \end{bmatrix} \\ 0 & \psi_1 \end{bmatrix} \sim \begin{bmatrix} y & \pi & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ 0 & 0 & y^2 & \pi \\ 0 & 0 & \pi^3 & -y \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ -y\pi & -\pi^2 & y^2 & \pi \\ y^2 & y\pi & \pi^3 & -y \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & -1 \\ \pi^3 & -y^2 & y\pi^2 & 0 \\ -y\pi & -\pi^2 & y^2 & 0 \\ y^2 & y\pi & \pi^3 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi^3 & y^2 & y\pi^2 \\ 0 & y\pi & -\pi^2 & y^2 \\ 0 & y^2 & -y\pi & -\pi^3 \end{bmatrix}$$

Since this is the presenting matrix for A, the AR sequence ending in N_1 is $0 \to M_1 \to A \to N_1 \to 0$. As always, we take syzygies to see that the AR sequence ending in M_1 is $0 \to N_1 \to B \oplus R \to M_1 \to 0$.

Now we compute the AR sequence ending in $M_2 \cong (y^2, \pi^2)R$. The resolution of M_2 starts out $0 \to M_2 \to R^2 \to M_2 \to 0$. Pull back along the endomorphism of M_2 given by multiplication by y^2. This map factors through R^2, using $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for the splitting map $M_2 \to R^2$. Similarly, the map $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi x \end{smallmatrix}\right)$ factors the map given by multiplication by $-\pi^3$ through R^2.

These two maps are πg and yg, respectively, where g is the endomorphism of M_2 given by multiplication by $-\pi^3/y$. We will show that g gives the AR sequence ending in M_2. The map g has associated matrix factorization $(\gamma, \delta) = \left(\left[\begin{smallmatrix} 0 & \pi \\ -y\pi & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ \pi & 0 \end{smallmatrix}\right]\right)$, so the middle term of the extension obtained by pulling back via g has presenting matrix $\eta = \left[\begin{smallmatrix} \psi_2 & \delta \\ 0 & \varphi_2 \end{smallmatrix}\right]$. Since this extension does not split, g gives the socle element of $\operatorname{Ext}^1_R(M_2, M_2)$, and the AR sequence ending in M_2 is $0 \to M_2 \to X \to M_2 \to 0$. (Recall that $X \cong Y$.)

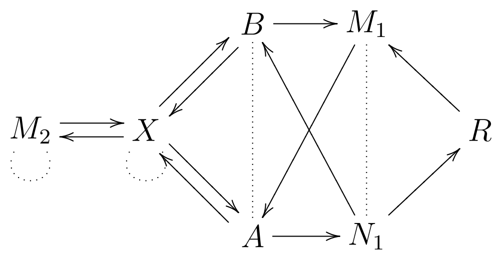
To finish the AR quiver, we use a process of elimination and count ranks. Consider the AR sequence ending in A. The third term, A, has rank two and its first syzygy, B, has rank one. The middle term, then, must have rank three. It has a summand isomorphic to M_1, and the complement U must have rank two. Considering the AR sequence for B, we see that the first syzygy of U must have rank two as well. The

AR sequences for N_1 and M_2 don't involve A, so U has no summand isomorphic to either of those. The only option left is $U \cong X$. This gives the AR sequence ending in A: $0 \to B \to X \oplus M_1 \to A \to 0$. The AR sequence ending in B is, taking first syzygies, $0 \to A \to X \oplus N_1 \to B \to 0$.

Finally, the middle term of the AR sequence ending in X must have rank four, and involves summands isomorphic to A, B, and M_2. Hence the AR sequence is

$$0 \to X \to A \oplus B \oplus M_2 \to X \to 0.$$

The complete AR quiver is then as follows. We again use Theorem 3.1.8 to conclude that it is indeed the whole quiver.



(E_6)

The (E'_6) singularity

Let $R = V[[y]]/(\pi^3 + y^4)$, and assume that the residue field characteristic p is greater than 2. This case is exactly symmetric to the (E_6) case just completed. So R has finite CM type.

The (E_7) singularity

Let $R = V[[y]]/(y^3 + y\pi^3)$. Assume that the residue field characteristic p is not equal to 2. Define seven pairs of matrices over R:

$$\begin{aligned} \alpha &= [y] & \beta &= [y^2+\pi^3] \\ \gamma &= \begin{bmatrix} y^2 & y\pi \\ y\pi^2 & -y^2 \end{bmatrix} & \delta &= \begin{bmatrix} y & \pi \\ \pi^2 & -y \end{bmatrix} \\ \varphi_1 &= \begin{bmatrix} y & \pi \\ y\pi^2 & -y^2 \end{bmatrix} & \psi_1 &= \begin{bmatrix} y^2 & \pi \\ y\pi^2 & -y \end{bmatrix} \\ \varphi_2 &= \begin{bmatrix} y & \pi^2 \\ y\pi & -y^2 \end{bmatrix} & \psi_2 &= \begin{bmatrix} y^2 & \pi^2 \\ y\pi & -y \end{bmatrix} \\ \chi_1 &= \begin{bmatrix} y\pi^2 & -y^2 & -y^2\pi \\ y\pi & \pi^2 & -y^2 \\ y^2 & y\pi & y\pi^2 \end{bmatrix} & \eta_1 &= \begin{bmatrix} \pi & 0 & y \\ -y & y\pi & 0 \\ 0 & -y & \pi \end{bmatrix} \\ \chi_2 &= \begin{bmatrix} y^2 & -\pi^2 & -y\pi \\ y\pi & y & -\pi^2 \\ y\pi^2 & y\pi & y^2 \end{bmatrix} & \eta_2 &= \begin{bmatrix} y & 0 & \pi \\ -y\pi & y^2 & 0 \\ 0 & -y\pi & y \end{bmatrix} \\ \chi_3 &= \begin{bmatrix} y^2 & y\pi & \pi & 0 \\ y\pi^2 & -y^2 & 0 & \pi \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^2 & -y \end{bmatrix} & \eta_3 &= \begin{bmatrix} y & \pi & -\pi & 0 \\ \pi^2 & -y & 0 & -\pi \\ 0 & 0 & y^2 & y\pi \\ 0 & 0 & y\pi^2 & -y^2 \end{bmatrix} \end{aligned}$$

As usual, we put

$$\begin{aligned} A &= \operatorname{coker}\alpha, & B &= \operatorname{coker}\beta \\ C &= \operatorname{coker}\gamma, & D &= \operatorname{coker}\delta \\ M_j &= \operatorname{coker}\varphi_j, & N_j &= \operatorname{coker}\psi_j \\ X_j &= \operatorname{coker}\chi_j, & Y_j &= \operatorname{coker}\eta_j. \end{aligned} \tag{3.2.11}$$

It is a fun exercise to figure out the multiplicities of these modules. This will be useful later. We obtain

$$\begin{aligned} & e(A) = 1 \qquad e(B) = 2 \\ & e(C) = 4 \qquad e(D) = 2 \\ & e(M_1) = e(N_1) = e(M_2) = e(N_2) = 3 \\ & e(X_1) = 6 \qquad e(Y_1) = 3 \\ & e(X_2) = 5 \qquad e(Y_2) = 4 \\ & e(X_3) = 6 \qquad e(Y_3) = 6. \end{aligned} \tag{3.2.12}$$

The ring itself clearly has multiplicity 3. Let's compute the AR sequence ending in A. The beginning of a resolution of A is $0 \to B \to R \to A \to 0$. Since $A \cong R/(y)$, pulling back along the map given by multiplication by y certainly splits this exact sequence. Since y^2 kills A, we see that $\pi^3 A = (y^2 + \pi^3)A$. Therefore the map on A given by multiplication by π^3 factors through R by sending $x \in A$ to $x \in R$, which then goes to $(y^2 + \pi^3)x = \pi^3 x$. We will show that the socle of $\operatorname{Ext}^1_R(A, B)$ is generated by the image of $\pi^2 \in \operatorname{Hom}_R(A, A)$. A presentation matrix for the middle term of the exact sequence obtained from π^2 is $\left[\begin{smallmatrix} y^2+\pi^3 & \pi^2 \\ 0 & y \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} y^2 & \pi^2 \\ y\pi & -y \end{smallmatrix}\right]$, which is the presenting matrix for N_2. This exact sequence does not split, so the AR sequence ending in A is thus $0 \to B \to N_2 \to A \to 0$. Taking syzygies gives the AR sequence ending in B: $0 \to A \to M_2 \to B \to 0$.

Next consider the AR sequence ending in $D \cong (y^2, y\pi)R$. We will show that the image of π generates the socle of $\operatorname{Ext}^1_R(D, C)$. Multiplication by π^2 on D factors through R^2 via the map $D \to R^2$ given by $x \mapsto \left(\begin{smallmatrix} 0 \\ -x/\pi^2 \end{smallmatrix}\right)$. Similarly, multiplication by y on D admits a factorization $x \mapsto \left(\begin{smallmatrix} -x/\pi^3 \\ 0 \end{smallmatrix}\right)$ through the free module R^2. Thus both these maps give the zero element of $\operatorname{Ext}^1_R(D, C)$.

Now, pulling back by π gives a nonsplit extension with middle term

$$\chi_3 = \begin{bmatrix} y^2 & y\pi & \pi & 0 \\ y\pi^2 & -y^2 & 0 & \pi \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^2 & -y \end{bmatrix},$$

so the AR sequence ending in D is $0 \to C \to X_3 \to D \to 0$. Taking syzygies shows that the AR sequence ending in C is $0 \to D \to Y_3 \to C \to 0$.

Next we compute the AR sequence ending in $M_1 \cong (y^2, \pi)R$. Consider the endomorphism h of M_1 given by multiplication by y^2/π. Then yh is multiplication by $-y\pi^2$, and πh is multiplication by y^2, both of which split a free resolution of M_1, as we shall show.

The map on M_1 given by $-y\pi^2$ factors through R^2 via $x \mapsto \left(\begin{smallmatrix} 0 \\ -y\pi x \end{smallmatrix}\right)$, so gives the zero element of $\operatorname{Ext}^1_R(M_1, N_1)$. The map given by multiplication by y^2 admits a factorization $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ through R^2.

So both of the maps yh and πh result in split exact sequences. All that remains is to show that h does not split a resolution of M_1. The map h is represented by the pair of matrices $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -y\pi^2 & 0 \end{smallmatrix}\right]$, $\beta = \left[\begin{smallmatrix} 0 & -y \\ y\pi & 0 \end{smallmatrix}\right]$. A presenting matrix for the middle term of the extension obtained by pulling back by h is then

$$\begin{bmatrix} y^2 & \pi & 0 & -y \\ y\pi^2 & -y & y\pi & 0 \\ 0 & 0 & y & \pi \\ 0 & 0 & y\pi^2 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \pi & 0 & y \\ 0 & -y & y\pi & 0 \\ 0 & 0 & -y & \pi \end{bmatrix}.$$

This is the presenting matrix for $R \oplus Y_1$, so the extension obtained by pulling back by h is not split. Hence the middle term of the AR sequence ending in M_1 is $R \oplus Y_1$, and the AR sequence is $0 \to N_1 \to R \oplus Y_1 \to M_1 \to 0$. Taking syzygies give the AR sequence ending in N_1, $0 \to M_1 \to X_1 \to N_1 \to 0$.

Don't stop now: on we go to $M_2 \cong (y^2, \pi^2)R$. Define a map h on M by multiplication by y^2/π. Then yh is multiplication by $-y\pi^2$, and πh is multiplication by y^2. The map given by multiplication by $-y\pi^2$ admits the factorization $x \mapsto \left(\begin{smallmatrix} 0 \\ -yx \end{smallmatrix}\right)$ through the free module R^2. Similarly, the map $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ from M_2 to R^2 factors multiplication by y^2. To show that h goes to the socle element of $\operatorname{Ext}^1_R(M_2, N_2)$, then, we need only show that h does not split a free resolution of M_2. A matrix factorization representing h is $\left(\left[\begin{smallmatrix} 0 & \pi \\ -y^2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ y & 0 \end{smallmatrix}\right]\right)$, and so the middle term of the sequence obtained from h is presented by the matrix

$$\begin{bmatrix} y^2 & \pi^2 & 0 & -y\pi \\ y\pi & -y & y & 0 \\ 0 & 0 & y & \pi^2 \\ 0 & 0 & y\pi & -y^2 \end{bmatrix} \sim \begin{bmatrix} y & 0 & 0 & 0 \\ 0 & y^2 & -\pi^2 & -y\pi \\ 0 & y\pi & y & -\pi^2 \\ 0 & y\pi^2 & y\pi & y^2 \end{bmatrix}.$$

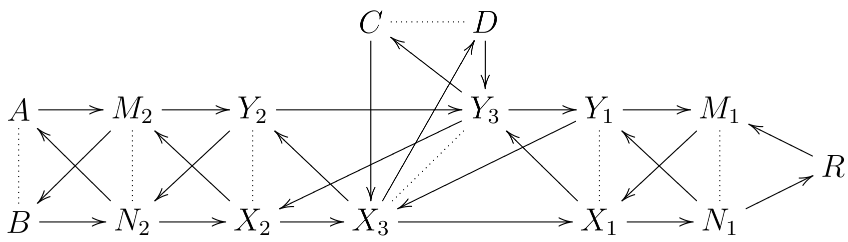
This gives the AR sequence ending in M_2, $0 \to N_2 \to A \oplus X_2 \to M_2 \to 0$, and the AR sequence ending in N_2 by taking syzygies: $0 \to M_2 \to B \oplus Y_2 \to N_2 \to 0$.

In order to compute the rest of the AR sequences, we refer to the multiplicities calculated earlier. Consider the AR sequence ending in Y_3. Since Y_3 and its first syzygy, X_3, both have multiplicity 6, the middle term of the AR sequence must have multiplicity 12. We know that is isomorphic to $D \oplus U$ for some U, which must have multiplicity 10 and not involve any of the other modules we have considered so far. Furthermore, its first syzygy must have also have multiplicity 10. It is an easy process of elimination to see that the only possibility is $U \cong X_1 \oplus Y_2$. Hence the AR sequence ending in Y_3 is $0 \to X_3 \to D \oplus X_1 \oplus Y_2 \to Y_3 \to 0$. Taking syzygies gives the AR sequence ending in X_3: $0 \to Y_3 \to C \oplus Y_1 \oplus X_2 \to X_3 \to 0$.

Next consider the AR sequence ending in X_1. The middle term must have multiplicity 9. We already have arrows to X_1 from X_3 and M_1, and $M_1 \oplus X_3$ has multiplicity 9. This gives the AR sequence $0 \to Y_1 \to M_1 \oplus X_3 \to X_1 \to 0$. Similar reasoning gives the AR sequence ending in Y_1: $0 \to X_1 \to N_1 \oplus Y_3 \to Y_1 \to 0$.

Finally, the middle term of the AR sequence ending in X_2 has multiplicity 9, and has direct summands isomorphic to N_2 and Y_3, so is isomorphic to $N_2 \oplus Y_3$. Taking syzygies for the AR sequence ending in Y_2, we get $0 \to Y_2 \to N_2 \oplus Y_3 \to X_2 \to 0$ and $0 \to X_2 \to M_2 \oplus X_3 \to Y_2 \to 0$.

This completes a connected component of the AR quiver for R, and so we have the whole quiver by Theorem 3.1.8.



(E_7)

The ($\mathbf{E}_7'$) singularity

Set $R = V[[y]]/(\pi^3 + y^3\pi)$. The matrix calculations of the previous section remain valid with y and π interchanged, but since π is a zerodivisor in R, we cannot apply Theorem 3.1.8 to conclude that R has finite CM type.

The ($\mathbf{E}_8$) singularity

Let $R = V[[y]]/(y^3 + \pi^5)$ with $p \neq 3$, a simple singularity of type (E_8). Define matrices over R:

$$\varphi_1 = \begin{bmatrix} y & \pi \\ \pi^4 & -y^2 \end{bmatrix} \qquad \psi_1 = \begin{bmatrix} y^2 & \pi \\ \pi^4 & -y \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} y & \pi^2 \\ \pi^3 & -y^2 \end{bmatrix} \qquad \psi_2 = \begin{bmatrix} y^2 & \pi^2 \\ \pi^3 & -y \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} \pi & -y & 0 \\ 0 & \pi & -y \\ y & 0 & \pi^3 \end{bmatrix} \qquad \beta_1 = \begin{bmatrix} \pi^4 & y\pi^3 & y^2 \\ -y^2 & \pi^4 & y\pi \\ -y\pi & -y^2 & \pi^2 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} \pi & -y & 0 \\ 0 & \pi^2 & -y \\ y & 0 & \pi^2 \end{bmatrix} \qquad \beta_2 = \begin{bmatrix} \pi^4 & y\pi^2 & y^2 \\ -y^2 & \pi^3 & y\pi \\ -y\pi^2 & -y^2 & \pi^3 \end{bmatrix}$$

$$\gamma_1 = \begin{bmatrix} \pi & y & 0 & \pi^3 \\ y & 0 & -\pi^3 & 0 \\ -\pi^3 & 0 & -y^2 & 0 \\ 0 & -\pi^2 & -y\pi & -y^2 \end{bmatrix} \qquad \delta_1 = \begin{bmatrix} 0 & y^2 & -\pi^3 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 \\ 0 & -\pi^2 & -y & 0 \\ \pi^2 & 0 & \pi & -y \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} \varphi_2 & \begin{pmatrix} 0 & \pi \\ -y\pi^2 & 0 \end{pmatrix} \\ 0 & \psi_2 \end{bmatrix} \qquad \delta_2 = \begin{bmatrix} \psi_2 & \begin{pmatrix} 0 & y\pi \\ -\pi^2 & 0 \end{pmatrix} \\ 0 & \varphi_2 \end{bmatrix}$$

$$\chi_1 = \begin{bmatrix} \beta_2 & \begin{pmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{pmatrix} \\ 0 & \alpha_2 \end{bmatrix} \qquad \eta_1 = \begin{bmatrix} \alpha_2 & \begin{pmatrix} 0 & 0 & -y \\ y\pi & 0 & 0 \\ 0 & y\pi & 0 \end{pmatrix} \\ 0 & \beta_2 \end{bmatrix}$$

$$\chi_2 = \begin{bmatrix} \pi^4 & y^2 & 0 & -y\pi^2 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 & 0 \\ 0 & -\pi^2 & -y & 0 & \pi^3 \\ -y\pi^2 & \pi^3 & 0 & y^2 & 0 \\ -\pi^3 & 0 & -\pi^2 & y\pi & -y^2 \end{bmatrix} \qquad \eta_2 = \begin{bmatrix} \pi & -y & 0 & 0 & 0 \\ y & 0 & 0 & \pi^2 & 0 \\ -\pi^2 & 0 & -y^2 & 0 & -\pi^3 \\ 0 & -\pi^2 & 0 & y & 0 \\ 0 & 0 & \pi^2 & \pi & -y \end{bmatrix}$$

As always, we associate modules to these matrices by $M_j = \operatorname{coker}\varphi_j$, $N_j = \operatorname{coker}\psi_j$, $A_j = \operatorname{coker}\alpha_j$, $B_j = \operatorname{coker}\beta_j$, $C_j = \operatorname{coker}\gamma_j$, $D_j = \operatorname{coker}\delta_j$, $X_j = \operatorname{coker}\chi_j$, $Y_j = \operatorname{coker}\eta_j$ for $j = 1, 2$.

Some of these are ideals: $M_1 \cong (y^2, \pi)R$, $M_2 \cong (y^2, \pi^2)R$, $N_1 \cong (y, \pi)R$, $N_2 \cong (y, \pi^2)R$, $A_1 \cong (y^2, y\pi^3, \pi^4)R$, and $A_2 \cong (y^2, y\pi^2, \pi^4)R$. The C_i and D_i all have rank 2, as does Y_2. The remaining modules, X_1, X_2, and Y_1, have rank 3.

Let's compute the AR sequences ending in these modules, starting with $M_1 = (y^2, \pi)R$. Starting from a free presentation of M_1, $0 \to N_1 \to R^2 \to M_1 \to 0$, pull back along the endomorphisms of M_1 given by multiplication by $-\pi^4$ and by y^2 to get split extensions. Multiplication by y^2 factors through R^2 via $x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$, while multiplication by $-\pi^4$ factors through R^2 with $x \mapsto \left(\begin{smallmatrix} 0 \\ -\pi^3 x \end{smallmatrix}\right)$.

The maps y^2 and $-\pi^4$ are πh and yh, respectively, where h is the endomorphism of M_1 defined by multiplication by y^2/π. We can show that h is in the socle of $\mathrm{Ext}^1_R(M_1, N_1)$, and hence gives the AR sequence ending in M_1. Write $h = \mathrm{coker}(\alpha, \beta)$, where $\alpha = \left[\begin{smallmatrix} 0 & 1 \\ -y\pi^3 & 0 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 0 & -y \\ \pi^3 & 0 \end{smallmatrix}\right]$. To identify the middle term of the extension given by h, consider

$$\begin{bmatrix} y^2 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ 0 & 0 & y & \pi \\ 0 & 0 & \pi^4 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ y\pi & 0 & y & \pi \\ \pi^5 & 0 & \pi^4 & -y^2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi & 0 & -y \\ \pi^4 & -y & \pi^3 & 0 \\ y\pi & 0 & y & \pi \\ 0 & y\pi & 0 & -y^2 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & 0 & -y \\ 0 & -y & \pi^3 & 0 \\ 0 & 0 & y & \pi \\ 0 & y\pi & 0 & -y^2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & \pi & 0 & y \\ 0 & -y & \pi^3 & 0 \\ 0 & y & \pi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & \pi & -y & 0 \\ 0 & 0 & \pi & -y \\ 0 & y & 0 & \pi^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the defining matrix for $A_1 \oplus R$, so we get the AR sequence ending in M_1: $0 \to N_1 \to A_1 \oplus R \to M_1 \to 0$. Taking syzygies implies that the AR sequence ending in N_1 is $0 \to M_1 \to B_1 \to N_1 \to 0$.

Now consider the first part of a resolution of $M_2 \cong (y^2, \pi^2)R$: $0 \to N_2 \to R^2 \to M_2 \to 0$. Pull back by the map given by multiplication by π^4 on M_2. This admits a factorization $x \mapsto \left(\begin{smallmatrix} 0 \\ \pi^2 x \end{smallmatrix}\right)$ through R^2, so results in a split sequence. Similarly, the map on M_1 given by multiplication by $-y^2$ factors through R^2 with $x \mapsto \left(\begin{smallmatrix} -x \\ 0 \end{smallmatrix}\right)$.

Let h be the endomorphism of M_2 given by multiplication by π^4/y. Then yh is multiplication by π^4 and πh is multiplication by $-y^2$ on M_2. Both of these endomorphisms give split exact sequences, so if h does not give a split sequence, then h gives the socle element of $\mathrm{Ext}^1_R(M_2, N_2)$. To identify the middle term of the sequence obtained by pulling back by h, note that a matrix factorization for h is $\left(\left[\begin{smallmatrix} 0 & \pi \\ -y\pi^3 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -y\pi \\ -\pi^2 & 0 \end{smallmatrix}\right]\right)$, so the middle term of the sequence is given by the matrix δ_2, which is a presentation matrix for D_2, and so the AR sequence is $0 \to N_2 \to D_2 \to M_2 \to 0$. Again, syzygies give us that the AR sequence ending in N_2 is $0 \to M_2 \to C_2 \to N_2 \to 0$.

Moving along, we consider a free resolution of A_1: $0 \to B_1 \to R^3 \to A_1 \to 0$. Let h be the endomorphism of A_1 given by multiplication by $y^2\pi$; then yh is multiplication by $-\pi^4$ and πh is multiplication by y^2. Both of these maps factor through the free module R^3, via

$$x \mapsto \begin{pmatrix} 0 \\ 0 \\ -x \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix},$$

respectively. Thus both πh and yh induce split sequences. We can write

$$h = \operatorname{coker}\left(\begin{bmatrix} 0 & 0 & \pi^3 \\ -\pi & 0 & 0 \\ 0 & \pi^3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -y\pi^2 \\ y & 0 & 0 \\ 0 & y & 0 \end{bmatrix}\right)$$

A matrix computation shows that

$$\begin{bmatrix} \pi^4 & y\pi^3 & y^2 & 0 & 0 & -y\pi^2 \\ -y^2 & \pi^4 & y\pi & y & 0 & 0 \\ -y\pi & -y^2 & \pi^2 & 0 & y & 0 \\ 0 & 0 & 0 & \pi & -y & 0 \\ 0 & 0 & 0 & 0 & \pi & -y \\ 0 & 0 & 0 & y & 0 & \pi^3 \end{bmatrix} \sim \begin{bmatrix} 0 & y^2 & -\pi^3 & 0 & 0 & 0 \\ -y^2 & y\pi & 0 & -\pi^3 & 0 & 0 \\ 0 & -\pi^2 & -y & 0 & 0 & 0 \\ \pi^2 & 0 & \pi & -y & 0 & 0 \\ 0 & 0 & 0 & 0 & y^2 & \pi^2 \\ 0 & 0 & 0 & 0 & \pi^3 & -y \end{bmatrix}$$

which is not split, so that the middle term of the AR sequence ending in A_1 is $N_1 \oplus D_1$. This gives the AR sequence $0 \to B_1 \to N_1 \oplus D_1 \to A_1 \to 0$. Taking syzygies gives the AR sequence ending in B_1: $0 \to A_1 \to M_1 \oplus C_1 \to B_1 \to 0$.

The computation for $A_2 \cong (y^2, y\pi^2, \pi^4)R$ is very similar. Let h be the endomorphism of A_2 given by multiplication by π^4/y. Then yh is multiplication by π^4 and πh is multiplication by $-y^2$ on A_2. These both give split sequences: The maps

$$x \mapsto \begin{pmatrix} -x \\ 0 \\ 0 \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

give factorizations of $-y^2$ and π^4, respectively, through R^3. We can compute the middle term of the exact sequence given by pulling back along h, and if it is nonsplit, we will have identified the AR sequence ending in A_2. Since we can write

$$h = \operatorname{coker}\left(\begin{bmatrix} 0 & 0 & -\pi^3 \\ \pi^2 & 0 & 0 \\ 0 & \pi^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{bmatrix}\right)$$

the middle term is given by the matrix

$$\chi_1 = \begin{bmatrix} \beta_2 & \begin{pmatrix} 0 & 0 & y\pi \\ -y & 0 & 0 \\ 0 & -y\pi & 0 \end{pmatrix} \\ 0 & \alpha_2 \end{bmatrix}$$

so the AR sequence ending in A_2 is $0 \to B_2 \to X_1 \to A_2 \to 0$. Taking syzygies gives the AR sequence ending in B_2: $0 \to A_2 \to Y_1 \to B_2 \to 0$.

In computing the rest of the AR sequences, we can use the process of elimination and compute ranks, since R is a domain. First consider the AR sequence ending in C_1. The first term is D_1, which has rank 2, so the middle term has rank 4. Since we already have an arrow $A_1 \to C_1$ in the AR quiver, we know that the middle term has a direct summand isomorphic to A_1. The complement has rank 3, and its first syzygy has rank 2 (since B_1 has rank 2). So the complement is either X_2 or a direct sum of B_1 and a rank-one module which has rank-one first syzygy. If the latter, then the rank-one would be one of M_1, M_2, N_1, or N_2. But we have already computed these AR sequences, and have no arrow from D_1 to any of these. So X_1 is the complement, and the AR sequence ending in C_1 is given by $0 \to D_1 \to A_1 \oplus X_2 \to C_1 \to 0$. Taking syzygies gives $0 \to C_1 \to B_1 \oplus Y_2 \to D_1 \to 0$, the AR sequence ending in D_1.

Next consider the AR sequence ending in C_2. Since C_2 and D_2 have rank 2, the middle term of the AR sequence has rank 4. We know that the AR quiver contains an arrow $M_2 \to C_2$, so M_2 is a summand of the middle term, leaving a rank-three complement with rank-three first syzygy. This complement contains none of the modules we have treated up to now, so must be one of X_1, X_2, Y_1, or Y_2. We know that Y_2 has the wrong rank, and the first syzygy of X_2 (that is, Y_2) also has the wrong rank. It can be checked that the map h on C_2 taking the generators (elements of R^4)

to

$$\begin{bmatrix} y\pi^4 \\ -y^2\pi^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -y^2\pi \\ -\pi^4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y^2\pi \\ \pi^4 \\ -y^2\pi^2 \end{bmatrix}, \begin{bmatrix} \pi^5 \\ 0 \\ -y^2\pi \\ -y\pi^4 \end{bmatrix}$$

satisfies $yh = \pi^4$ and $\pi h = -y^2$. Each of these splits a free presentation of C_2. The endomorphism of C_2 given by multiplication by y^2 factors through R^4 by sending the generators (columns of δ_2) to the columns of

$$\begin{bmatrix} 0 & -\pi^2 & 0 & 0 \\ -y\pi^3 & 0 & y\pi^2 & -\pi^4 \\ 0 & 0 & 0 & y\pi^2 \\ 0 & 0 & -\pi^3 & 0 \end{bmatrix}.$$

The endomorphism given by multiplication by π^4 factors through R^4 by sending the generators to the columns of the matrix

$$\begin{bmatrix} 0 & -y\pi & 0 & -y^2 \\ y^2\pi^2 & 0 & -y^2\pi & 0 \\ 0 & 0 & 0 & -y^2\pi \\ 0 & 0 & y\pi^2 & 0 \end{bmatrix}.$$

We can factor h as a pair of maps between free modules

$$h = \operatorname{coker}\left(\begin{bmatrix} 0 & 0 & 0 & -y \\ 0 & 0 & -y\pi & 0 \\ y^2 & \pi^2 & 0 & -y\pi \\ \pi^3 & -y & \pi^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y\pi & 0 & y \\ -\pi^2 & 0 & y\pi & 0 \\ y & \pi^2 & 0 & 2\pi \\ \pi^3 & -y^2 & -2y\pi^2 & 0 \end{bmatrix} \right)$$

Hence the middle term of the short exact sequence obtained by pulling back via h is presented by the matrix

$$\begin{bmatrix} y & \pi^2 & 0 & \pi & 0 & y\pi & 0 & y \\ \pi^3 & -y^2 & -y\pi^2 & 0 & -\pi^2 & 0 & y\pi & 0 \\ 0 & 0 & y^2 & \pi^2 & y & \pi^2 & 0 & 2\pi \\ 0 & 0 & \pi^3 & -y & \pi^3 & -y^2 & -2y\pi^2 & 0 \\ 0 & 0 & 0 & 0 & y^2 & \pi^2 & 0 & y\pi \\ 0 & 0 & 0 & 0 & \pi^3 & -y & -\pi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & \pi^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi^3 & -y^2 \end{bmatrix} \sim \begin{bmatrix} y & \pi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \pi^3 & -y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi^4 & y\pi^3 & y^2 & 0 & 0 & y \\ 0 & 0 & -y^2 & \pi^3 & y\pi & -y & 0 & 0 \\ 0 & 0 & -y\pi^2 & -y^2 & \pi^3 & 0 & -y\pi & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi^2 & - \\ 0 & 0 & 0 & 0 & 0 & y & 0 & \pi \end{bmatrix}$$

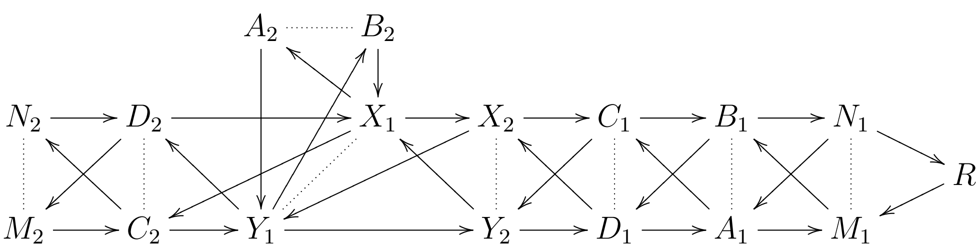
This is the presentation matrix for $M_2 \oplus X_1$, so the AR sequence ending in C_2 is given by $0 \to D_2 \to M_2 \oplus X_1 \to N_2 \to 0$. Taking syzygies gives the AR sequence ending in D_2: $0 \to C_2 \to N_2 \oplus Y_1 \to D_2 \to 0$.

The middle term of the AR sequence ending in X_1 has rank 6, and has summands of B_2 and D_2 from existing arrows in our quiver. The only other rank-two is Y_2. (Besides, we know the complement must have rank 2 and a rank-three syzygy, so must be Y_2.) This gives the two AR sequences $0 \to Y_1 \to B_2 \oplus D_2 \oplus Y_2 \to X_1 \to 0$ and $0 \to X_1 \to A_2 \oplus C_2 \oplus X_2 \to Y_1 \to 0$.

Finally consider the AR sequence ending in X_2. Since X_2 has rank 3 and Y_2 has rank 2, the middle term has rank 5. We already have an arrow $D_1 \to X_2$ and an arrow $X_1 \to X_2$, so the middle term is $D_1 \oplus X_1$, and the AR sequence is

$$0 \to Y_2 \to D_1 \oplus X_1 \to X_2 \to 0.$$

Applying Theorem 3.1.8 shows that the AR quiver for R is as follows.



(E$_8$)

The ($\mathbf{E}_8'$) singularity

Again, symmetry saves the day. Let $R = V[[y]]/(\pi^3 + y^5)$, with residue field characteristic $p \neq 5$. Then R has finite CM type.

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