Epsilon-delta (ϵ–δ) proofs are used when we want to prove a limit statement, such as

(1) $$\lim_{x \to 2}(7x - 3) = 11.$$

Intuitively, we say that this limit statement is true "because" as x gets closer and closer to 2, the value of the function $7x - 3$ gets closer and closer to 11.[1]

However, this doesn't *prove* that (1) is true.

(It's worth pointing out here that we're using the verb "to prove" and the noun "proof" in the mathematical sense, which is pretty much completely different from the normal English usage. In the rest of your life, you might talk about proving that smoking is bad for you, or that aliens exist. In mathematics, a proof is a very particular thing: a sequence of statements showing that one assertion *logically follows* from another.)

Remember that in order to prove (1), we must show that

For any $\epsilon > 0$, there exists $\delta > 0$ so that any x satisfying $0 < \|x - 2\| < \delta$ also satisfies $\|(7x - 3) - 11\| < \epsilon$.

This is a little different from the usual wording, so read it carefully. Say it out loud. I've put in line breaks where you should pause.

What this says is that given any positive number ϵ, we have to find a positive number δ so that whenever the distance between a number x and 2 is less than δ, the distance between $(7x - 3)$ and 11 is less than ϵ.

I like to think of this process as a game, or competition. Some evil genius hands us an ϵ, which might be incredibly small. Our job is to take that ϵ and find a δ (*which will probably depend on which epsilon we were handed*) that is so super-duper-small that if x is within δ of 2, then $(7x - 3)$ is within the evil genius's ϵ of 11.

Finding Delta in terms of Epsilon. As I said above, once the evil genius hands us ϵ, we have to come up with a δ, and our δ will probably depend on what ϵ we were given. It's not at all clear from the beginning what this δ should be, but usually we can manipulate the quantity $|(7x - 3) - 11|$ to figure out one that will work. Consider this:

$$|(7x - 3) - 11| = |7x - 3 - 11| = |7x - 14| = 7|x - 2|.$$

Hurrah! What have we discovered? We've found that the quantity we're trying to make smaller than the evil genius's ϵ, that is, $|(7x - 3) - 11|$, is equal to 7 times the quantity we have control over, $|x - 2|$. In other words, we are going to *assume*, once we start the proof, that $|x - 2| < \delta$. That gives us control. Since we're going to *assume* that $|x - 2| < \delta$, we see that

$$|(7x - 3) - 11| = \cdots = 7|x - 2| < 7\delta.$$

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[1] Note that when we talk about x getting "close to" 2, we don't mean x is 1 or 3 – we're talking about things like 1.9999999 and 2.0000001 here, and things even closer to 2 than that.

What we wanted was to make $|(7x-3)-11|$ less than the given ϵ. What we managed to show was that $|(7x-3)-11|$ was less than 7δ. So what should we choose δ to be?

If we choose $\delta = \epsilon/7$ (note that this depends on ϵ, as we expected), then the calculation above becomes

$$|(7x-3)-11| = \cdots = 7|x-2| < 7\delta = \epsilon$$

and so we have shown the following statement: *If* $|x-2|$ is less than $\delta = \epsilon/7$ (but not zero), *then* $|(7x-3)-11| < \epsilon$. Exactly what we wanted!

Sometimes it helps to have concrete numbers to think about. Suppose that the evil genius has handed us $\epsilon = 1/3$. He's challenged us to find a δ so that for any x satisfying $0 < |x-2| < \delta$, it's also true that $|(7x-3)-11| < 1/3$. We can respond by taking $\delta = (1/3)/7 = 1/21$. If, on the other hand, the EG hands us $\epsilon = 1/10$, we can fight back with $\delta = 1/70$.

Writing Up a Delta-Epsilon Proof. We've now convinced ourselves that the limit statement (1) is true. Whatever little ϵ the EG throws at us, we can just divide it by 7 and take that for δ. What's left?

The last step is convincing someone else. This is a different process altogether. We've already done all the hard work – all that's left is to explain what we found. We're going to write down our ideas clearly, concisely, and using standard mathematical symbols and phrases. (This means no evil geniuses.) Remember that we're trying to prove that

For any $\epsilon > 0$, there exists a $\delta > 0$ so that
any x satisfying $0 < |x-2| < \delta$ also satisfies $|(7x-3)-11| < \epsilon$.

Our proof is going to follow the structure of that statement quite closely. First, we'll let ϵ be an arbitrary positive number. Then we'll give our δ (usually in terms of ϵ, like above). Then we'll demonstrate that this value of δ does what we asked of it: we'll *assume* that $0 < |x-2| < \delta$, make a series of steps, and finally *conclude* that $|(7x-3)-11| < \epsilon$.

The format is always the same. Here is how it looks for our example.

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ so that if $0 < |x-2| < \delta$, then $|(7x-3)-11| < \epsilon$. Take $\delta = \epsilon/7$.

If $0 < |x-2| < \delta$, then

$$\begin{aligned} |(7x-3)-11| &= |7x-3-11| \\ &= |7x-14| \\ &= 7|x-2| \\ &< 7\delta && \text{by assumption!} \\ &= 7(\epsilon/7) && \text{by our choice of } \delta \\ &= \epsilon \end{aligned}$$

So we have $|(7x-3)-11| < \epsilon$. □

Note that we state our intentions in the second sentence, so the reader knows what is coming. Then we "take" δ in the third sentence of the proof, even though we haven't yet shown the calculation that explains why this is the right choice. (You could also say "Put" or "Choose" or "Let".) When you're writing up an $\epsilon - \delta$ proof, you should do the scratchwork like we did on the first page to find out which δ works, then write up the proof like I did just above.