11

Semigroups of modules

chap:semigroups

In this section we study the different ways in which a finitely generated module can be decomposed as a direct sum of indecomposable modules. Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathscr{C}$ a full subcategory of the category R-mod of finitely generated R-modules. We assume that $\mathscr{C}$ is closed under isomorphism, finite direct sums, and direct summands. There is a set $\mathrm{V}(\mathscr{C}) \subseteq \mathscr{C}$ of representatives; each element $M \in \mathscr{C}$ is isomorphic to exactly one element $[M] \in \mathrm{V}(\mathscr{C})$. We make $\mathrm{V}(\mathscr{C})$ into an additive semigroup in the obvious way: $[M] + [N] = [M \oplus N]$. This monoid encodes information about direct-sum decompositions in $\mathscr{C}$.

Suppose R is complete (in the $\mathfrak{m}$-adic topology). By the Krull-Remak-Schmidt theorem cor:completeKRS 1.8, each $M \in R-\mathrm{mod}$ is uniquely a direct sum of indecomposable modules (up to isomorphism and ordering of the summands). Therefore $\mathrm{V}(\mathscr{C}) \cong \mathbb{N}_0^{(I)}$, where $\mathbb{N}_0$ is the additive semigroup of non-negative integers and the index set I is the set of atoms of $\mathrm{V}(\mathscr{C}\}$, that is, the set of representatives $[N]$ of indecomposable objects N in $\mathscr{C}$.

For a general local ring R, we can exploit the semigroup homomorphism

$$j : \mathrm{V}(R-\mathrm{mod}) \longrightarrow \mathrm{V}(\widehat{R}-\mathrm{mod})$$

taking $[M]$ to $[\widehat{R} \otimes_R M]$. This homomorphism is injective by Corollary cor:guralnick-completion 6.4, and it follows that the semigroup $\mathrm{V}(R-\mathrm{mod})$ is *cancellative*: $x + z = y + z \implies x = y$. Since, in this section, we will deal only with local rings, all of our semigroups are tacitly assumed to be cancellative. We'll also assume they are written additively, with neutral element 0, and that $x + y = 0 \implies x = y = 0$.

The homomorphism j actually satisfies a much stronger condition. Recall that a *divisor homomorphism* is a semigroup homomorphism $j : \Lambda_1 \longrightarrow \Lambda_2$ such that, for all $x, y \in \Lambda_1$, $j(x) \mid j(y) \implies x \mid y$. (Recall that "$x \mid y$" means that there is an element z such that $x + z = y$.) Corollary 6.4 says that $j : R-\text{mod} \longrightarrow \widehat{R}-\text{mod}$ is a divisor homomorphism. In fact, this holds much more generally:

div **11.1 Theorem** ([HW09, Theorem 1.3]). *Let $R \longrightarrow S$ be a flat local homomorphism of Noetherian local rings. Then the map* $\mathrm{V}(R-\text{mod}) \longrightarrow \mathrm{V}(S-\text{mod})$ *taking* $[M]$ *to* $[S \otimes_R M]$ *is a divisor homomorphism.* □

Proof. Suppose M and N are finitely generated R-modules and that $S \otimes_R M \mid S \otimes_R N$. We want to show that $M \mid N$. By 6.2 it will be enough to show that $M/\mathfrak{m}^t M \mid N/\mathfrak{m}^t N$ for all $t \geqslant 1$. By passing to the flat local homomorphism $R/\mathfrak{m}^t \longrightarrow S/\mathfrak{m}^t S$, we may assume that R is Artinian. By 10.3, we know, at least, that

add (11.1.1) $$M \mid N^{(r)} \text{ for some positive integer } r.$$

By Corollary 1.8 (or Theorem 1.4 and Corollary 1.6) M is uniquely a direct sum of indecomposable modules, say $M = V_1 \oplus \cdots \oplus V_r$ with each V_i indecomposable. If $r = 1$, then 11.1.1 and Krull-Remak-Schmidt uniqueness imply that $M \mid N$. An easy induction argument (using direct-sum cancellation) completes the proof (cf. Exercise 11.15). □

11.2 Definition. A *Krull monoid* is a monoid that admits a divisor homomorphism into a free monoid.

Every finitely generated Krull monoid admits a divisor homomorphism into $\mathbb{N}_0^{(t)}$ for some positive integer t. Conversely, it follows easily from Dickson's Lemma (Exerciseex:clutter) that a monoid admitting a divisor homomorphism to $\mathbb{N}_0^{(t)}$ must be finitely generated.

Finitely generated Krull monoids are called *positive normal affine semigroups* in BH [BH93]. From BH [BH93, Exercise 6.1.10, p. 252], we obtain the following characterization of these monoids:

monoid **11.3 Proposition.** *The following conditions on a semigroup Λ are equivalent:*

1. *Λ is a finitely generated Krull monoid.*
2. *$\Lambda \cong G \cap \mathbb{N}^{(t)}$ for some positive integer t and some subgroup G of $\mathbb{Z}^{(t)}$. (In the terminology of* BH *[BH93], Λ is* full subsemigroup of $\mathbb{N}^{(t)}$.)
3. $\Lambda \cong W \cap \mathbb{N}^{(u)}$ for some positive integer u and some $\mathbb{Q}$-subspace W of $\mathbb{Q}^{(n)}$. (That is, Λ is an *expanded subsemigroup of* $\mathbb{N}^{(n)}$ *(cf.* BH *[BH93]).)*
4. *There exist positive integers m,n and an $m \times n$ matrix α over $\mathbb{Z}$ such that $\Lambda \cong \mathbb{N}^{(n)} \cap \ker(\alpha)$.*

Obviously, every expanded subsemigroup of $\mathbb{N}^{(t)}$ is also a full subsemigroup, but the converse can fail. For example, the semigroup $H := \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{N}_0^{(2)} \mid x \equiv y \pmod 3\}$ of is *not* the restriction to $\mathbb{N}_0^{(2)}$ of the kernel of a matrix. However, H is isomorphic to $H_1 := \{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{N}_0^{(3)} \mid x + 2y = 3z\}$. As this example indicates, the number n of (4) might be larger than the number t of (2).
Item (4) says that a finitely generated Krull monoid can be regarded as the collection of non-negative integer solutions of a homogeneous system

of linear equations. For this reason these monoids are sometimes called *Diophantine* monoids.

In order to study uniqueness of direct-sum decompositions, it's really enough to examine a small piece of the class R-mod of all finitely generated modules. Given a finitely generated module M, recall (Notation not:add 10.2) that $\mathrm{add}(M)$ denotes that category of modules that are isomorphic to direct summands of direct sums of finitely many copies of M. We note that $+(M) := \mathrm{V}(\mathrm{add}(M))$ is a finitely generated Krull monoid, since the divisor homomorphism $j : \mathrm{V}(R\,\mathrm{mod}) \longrightarrow \mathrm{V}(\widehat{R}\,\mathrm{mod})$ carries $+(M)$ into the free monoid generated by the isomorphism classes of the indecomposable direct summands of $\widehat{M}$. In the next two sections we will prove two realization theorems, which show that every finitely generated Krull monoid can be realized in the form $+(M)$, for a suitable local ring R and maximal CM module M.

§1 Realization theorem in dimension one

The key to understanding the monoids $\mathrm{V}(R-\mathrm{mod})$ and $+(M)$ is knowing which modules over the completion actually come from R-modules. More generally, if $R \longrightarrow S$ is a ring homomorphism, we say that the S-module N is *extended* (from R) provided there is an R-module M such that $S \otimes_R M \cong N$. In dimension one, a beautiful result due to Levy and Odenthal Levy-Odenthal:1996 [LO96] tells us exactly which $\widehat{R}$-modules are extended. First, we define, for any one-dimensional local ring $(R, \mathfrak{m}, k)$ the *Artinian localization* $\mathrm{K}(R)$ as follows:

$$\mathrm{K}(R) = (R - (P_1 \cup \cdots \cup P_s))^{-1} R,$$

where $P_1, \ldots, P_s$ are the minimal prime ideals of R (the prime ideals distinct from $\mathfrak{m}$). If R is Cohen-Macaulay, this is just the classical quotient ring. If R is *not* Cohen-Macaulay, the natural map $R \longrightarrow \mathrm{K}(R)$ is not one-to-one.

nextend **11.4 Theorem** ([LO96]). *Let $(R, \mathfrak{m}, k)$ be a one-dimensional local ring, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if* $\mathrm{K}(\widehat{R}) \otimes_{\widehat{R}} N$ *is extended from* $\mathrm{K}(R)$*.*

Proof. To simplify notation, we let $K = \mathrm{K}(R)$ and $L = \mathrm{K}(\widehat{R})$. If Q is a minimal prime ideal of $\widehat{R}$, then $Q \cap R$ is a minimal prime ideal of R, since "going down" holds for flat extensions [BH93, Lemma A.9]. Therefore the inclusion $R \longrightarrow S$ induces a homomorphism $K \longrightarrow L$, and this homomorphism is faithfully flat, since the map $\mathrm{Spec}(\widehat{R}) \longrightarrow \mathrm{Spec}(R)$ is surjective [BH93, Lemma A.10]. The "only if" direction is clear from the change-of-rings diagram:

$$\begin{array}{ccc} \widehat{R} & \longrightarrow & L \\ \uparrow & & \uparrow \\ R & \longrightarrow & K \end{array}$$

For the converse, let X be a finitely generated K-module such that $L \otimes_K X \cong L \otimes_S N$. Since K is a localization of R, there is a finitely generated R-module M such that $K \otimes_R M \cong X$. Since $L \otimes_S N \cong L \otimes_S (S \otimes_R M)$, there is a homomorphism $\alpha : N \longrightarrow S \otimes_R M$ inducing an isomorphism from $L \otimes_S N$ to $L \otimes_S (S \otimes_R M)$. Then the kernel U and cokernel V of α have finite length and therefore are extended. (Any $\widehat{R}$-module L of finite length also has finite length over R, and the natural map $L \longrightarrow \widehat{R} \otimes_R L$ is an isomorphism.) Now we break the exact sequence

$$0 \longrightarrow U \longrightarrow N \longrightarrow S \otimes_R M \longrightarrow V \longrightarrow 0$$

into two short exact sequences:

$$0 \longrightarrow U \longrightarrow N \longrightarrow W \longrightarrow 0$$

$$0 \longrightarrow W \longrightarrow S \otimes_R M \longrightarrow V \longrightarrow 0$$

Applying (2) of the next lemma to the second short exact sequence, we see that W is extended. Now we apply (1) of the lemma to the first short exact sequence, to conclude that N is extended. □

extended **11.5 Lemma.** *Let* $(R, \mathfrak{m})$ *be a local ring with completion* $(\widehat{R}, \widehat{m})$*, and let*

(11.5.1) $$0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

be an exact sequence of finitely generated $\widehat{R}$*-modules.*

1. *Assume* N' *and* N'' *are extended. If* $\operatorname{Ext}^1_{\widehat{R}}(N'', N')$ *has finite length as an* R*-module (e.g., if* N''_P *is* $\widehat{R}_P$*-free for each prime* $P \neq \widehat{m}$*), then* N *is extended.*

2. *Assume* N *and* N'' *are extended. If* $\operatorname{Hom}_{\widehat{R}}(N, N'')$ *has finite length as an* R*-module (e.g., if* ${}_{\widehat{R}}N''$ *has finite length), then* N' *is extended*

3. *Assume* N' *and* N *are extended. If* $\operatorname{Hom}_{\widehat{R}}(N', N)$ *has finite length as an* R*-module (e.g., if* ${}_{\widehat{R}}N'$ *has finite length), then* N'' *is extended.*

Proof. For (1), write $N' = S \otimes_R N'_0$ and $N'' = S \otimes_R N''_0$, where N'_0 and N''_0 are finitely generated left R-modules. The natural map $\widehat{R} \otimes_R \operatorname{Ext}^1_R(N''_0, N'_0) \longrightarrow$

$\operatorname{Ext}^1_{\widehat{R}}(N',N'')$ is an isomorphism By faithful flatness, $\operatorname{Ext}^1_R(N_0'',N_0')$ has finite length as an R-module. Therefore the natural map $\operatorname{Ext}^1_R(N_0'',N_0') \longrightarrow \widehat{R} \otimes_R \operatorname{Ext}^1_R(N_0'',N_0')$ is an isomorphism. Combining these two natural isomorphisms, we see that the given exact sequence, regarded as an element of $\operatorname{Ext}^1_{\widehat{R}}(N'',N')$, comes from a short exact sequence $0 \longrightarrow N_0' \longrightarrow N_0 \longrightarrow N_0'' \longrightarrow 0$. Clearly, then, $\widehat{R} \otimes_R N_0 \cong N$.

To prove (2), we write $N = \widehat{R}_R N_0$ and $N'' = \widehat{R} \otimes_R N_0''$, where N_0 and N_0'' are finitely generated left R-modules. As in the proof of (1) we see that the natural map $\operatorname{Hom}_R(N_0,N_0'') \longrightarrow \operatorname{Hom}_{\widehat{R}}(N,N')$ is an isomorphism. Therefore the $\widehat{R}$-homomorphism β comes from a homomorphism $\beta_0 \in \operatorname{Hom}_R(N_0,N_0'')$. Clearly, then, $N' \cong \widehat{R} \otimes_R \ker(\beta_0)$. The proof of (3) is essentially the same: Write $N = \widehat{R} \otimes_R N_0$ and $N' = S \otimes_R N_0'$. Show that α comes from some $\alpha_0 \in \operatorname{Hom}_R(N_0',N_0)$, and deduce that $N'' \cong S \otimes_R \operatorname{cok}(\alpha_0)$. □

extend **11.6 Corollary.** *Let $(R,\mathfrak{m},k)$ be a one-dimensional local ring whose completion $\widehat{R}$ is reduced, and let N be a finitely generated $\widehat{R}$-module. Then N is extended from R if and only if $\dim_{R_P}(N_P) = \dim_{R_Q}(N_Q)$ whenever P and Q are prime ideals of $\widehat{R}$ lying over the same prime ideal of R. In particular, if R is a domain, then N is extended if and only if N has constant rank.*

This gives us a strategy for producing strange direct-sum behavior:

1. Find a one-dimensional domain R whose completion has lots of minimal primes.
2. Build indecomposable $\widehat{R}$-modules with highly non-constant ranks.
3. Put them together in different ways to get constant-rank modules.

Suppose, for example, that R is a domain whose completion $\widehat{R}$ has two minimal primes P and Q. Suppose we can build indecomposable $\widehat{R}$-modules U, V, W and X, with ranks $(2,0), (0,2), (2,1)$ and $(1,2)$, respectively. Then $U \oplus V$ is extended, say, $U \oplus V \cong \widehat{M}$. Similarly, there are R-modules N, F and G such that $V \oplus W \oplus W \cong \widehat{N}, W \oplus X \cong \widehat{F}$ and $U \oplus X \oplus X \cong \widehat{G}$. Using the Krull-Remak-Schmidt theorem over $\widehat{R}$, we see easily that no non-zero proper direct summand of any of the modules $\widehat{M}, \widehat{N}, \widehat{F}, \widehat{G}$ has constant rank. It follows from Corollary 11.6 that M, N, F and G are indecomposable, and of course no two of them are isomorphic since (again by Krull-Remak-Schmidt) their completions are pairwise non-isomorphic. Finally, we see that $M \oplus F \oplus F \cong N \oplus G$, since the two modules have isomorphic completions. Thus we easily obtain a mild violation of Krull-Remak-Schmidt uniqueness over R.

It's easy to accomplish (1), getting a one-dimensional domain with a lot of splitting. In order to facilitate (2), however, we want to ensure that each analytic branch has infinite Cohen-Macaulay type. The following example from [Wie01] does the job nicely:

sigma **11.7 Example ([Wie01, (2.3)]).** Fix a positive integer s, and let k be any field with $|k| \geqslant s$. Choose distinct elements $t_1, \dots, t_s \in k$. Let Σ be the complement of the union of the maximal ideals $(X - t_i)k[X]$, $i = 1, \dots, s$. We define $R = R_s$ by the pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \Sigma^{-1}k[X] \\ \downarrow & & \downarrow \pi \\ k & \longrightarrow & \dfrac{\Sigma^{-1}k[X]}{(X-t_1)^4 \cdots\cdots (X-t_s)^4} \end{array}$$

where π is the natural map. Then R is a one-dimensional local domain, and $\widehat{R}$ is reduced with exactly s minimal prime ideals.

We remark that R is the ring of rational functions $f \in \mathbb{Q}(T)$ such that $f(t_1) = \cdots = f(t_s) \neq \infty$ and f', f'' and f''' vanish at each t_i.

Let $P_1, \ldots, P_s$ be the minimal prime ideals of $\widehat{R}$. By the *rank* of a finitely generated $\widehat{R}$-module N, we mean the s-tuple $(r_1, \ldots, r_s)$, where r_i is the dimension of N_{P_i} as a vector space over R_{P_i}. A jazzed-up version of the argument used to prove Theorem 2.5 yields the following:

rankN **11.8 Theorem** ([Wie01, (2.4)]). *Fix a positive integer s, and let $(r_1, \ldots, r_s)$ be any non-trivial sequence of non-negative integers. Then $\widehat{R_s}$ has an indecomposable maximal Cohen-Macaulay module N with* $\operatorname{rank}(N) = (r_1, \ldots, r_s)$. □

Thus even the case $s = 2$ of Example 11.7 yields the pathology discussed after Corollary 11.6.

Recalling (4) of Proposition 11.3, we say that the finitely generated Krull semigroup Λ *can be defined by m equations* provided $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$ for some n and some $m \times n$ integer matrix α. Given such an embedding of Λ in $\mathbb{N}_0^{(n)}$, we say a column vector $\lambda \in \Lambda$ is *strictly positive* provided each of its entries is a positive integer. By decreasing n (and removing some columns from α) if necessary, we can harmlessly assume (without changing m) that Λ contains a strictly positive element (cf. [WW09, Remark 3.1]).

CD **11.9 Corollary** ([Wie01, Theorem 2.1]). *Fix a non-negative integer m, and let R be the ring R_{m+1} of Example 11.7. Let Λ be a finitely generated Krull semigroup defined by m equations and containing a strictly positive element*

λ. Then there exist a maximal Cohen-Macaulay R-module M and a commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\subseteq} & \mathbb{N}_0^{(n)} \\ {\scriptstyle\varphi}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\psi}\big\downarrow{\scriptstyle\cong} \\ \mathrm{V}(+(M)) & \xrightarrow{i} & \mathrm{V}(+(\widehat{M})) \end{array},$$

in which

1. *i is the natural map taking $[F]$ to $[\widehat{F}]$,*

2. *φ and ψ are semigroup isomorphisms, and*

3. *$\varphi([M]) = \lambda$.*

Proof. We have $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$, where $\alpha = [a_{ij}]$ is an $m \times n$ matrix over $\mathbb{Z}$. Choose a positive integer h such that $a_{ij} \geqslant 0$ for all i, j. For $j = 1, \dots, n$, choose, using Theorem 11.8, a maximal Cohen-Macaulay $\widehat{R}$-module L_j such that $\operatorname{rank}(L_j) = (a_{1j} + h, \dots, a_{mj} + h, h)$.

Given any column vector $\beta = [b_1\ b_2\ \dots b_n]^{\mathrm{tr}} \in \mathbb{N}_0^{(n)}$, put $N_\beta = L_1^{(b_1)} \oplus \cdots \oplus L_n^{(b_n)}$. The rank of N_β is

$$\left(\sum_{j=1}^{n} (a_{1j} + h) b_j, \dots, \sum_{j=1}^{n} (a_{mj} + h) b_j, (\sum_{j=1}^{n} b_j) h \right).$$

Since R is a domain, Corollary 11.6 implies that N_β is in the image of $j : \mathrm{V}(R\,\mathrm{mod}) \longrightarrow \mathrm{V}(\widehat{R}\,\mathrm{mod})$ if and only if $\sum_{j=1}^{n} (a_{ij} + h) b_j = (\sum_{j=1}^{n} b_j) h$ for each i, that is, if and only if $\beta \in \mathbb{N}_0^{(n)} \cap \ker(\alpha) = \Lambda$. To complete the proof, we let M be the R-module (unique up to isomorphism) such that $\widehat{M} \cong N_\lambda$. $\square$

This corollary makes it very easy to demonstrate spectacular failure of Krull-Remak-Schmidt uniqueness:

11.10 Example. Let $\Lambda = \{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{N}_0^{(3)} \mid 72x + y = 73z\}$. This has three atoms (minimal non-zero elements), namely

$$\alpha := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \beta := \begin{bmatrix} 0 \\ 73 \\ 1 \end{bmatrix}, \quad \gamma := \begin{bmatrix} 73 \\ 0 \\ 72 \end{bmatrix}.$$

Note that $73\alpha = \beta + \gamma$, Taking $s = 2$ in Example 11.7, we get a local ring R and indecomposable R-modules M, F, G such that $M^{(t)}$ has only the obvious direct-sum decompositions for $t \leqslant 72$, but $M^{(73)} \cong F \oplus G$.

We define the *splitting number* $\text{spl}(R)$ of a one-dimensional local ring R by

$$\text{spl}(R) = |\text{Spec}(\widehat{R})| - |\text{Spec}(R)|.$$

The splitting number of the ring R_s in Example 11.7 is $s-1$. Corollary 11.9 says that every finitely generated Krull monoid defined by m equations can be realized as $+(M)$ for some finitely generated module over a one-dimensional local ring (in fact, a domain essentially of finite type over $\mathbb{Q}$) with splitting number m. This is the best possible:

expanded **11.11 Theorem.** *Let M be a finitely generated module over a one-dimensional local ring R with splitting number m. The embedding* $+(M) hookrightarrow \text{V}(\widehat{R}-$mod) *exhibits* $+(M)$ *has an expanded subsemigroup of the free semigroup* $+(\widehat{M})$*. Moreover,* $+(M)$ *is defined by m equations.*

Proof. Write $\widehat{M} = V_1^{(e_1)} \oplus \cdots \oplus V_n^{(e_n)}$, where the V_j are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the e_i are all positive. We have an embedding $+(M) \hookrightarrow \mathbb{N}_0^{(n)}$ taking $[F]$ to $[b_1 \ \ldots b_n]^{\text{tr}}$, where $\widehat{F} \cong V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$, and we identify $+(M)$ with its image Λ in $\mathbb{N}_0^{(n)}$. Given a prime $P \in \operatorname{Spec}(R)$ with, say, t primes $Q_1, \dots, Q_t$ lying over it, there are $t-1$ homogeneous linear equations on the b_j that say that $\widehat{N}$ has constant rank on the fiber over P (cf. Corollary 11.6). Letting P vary over $\operatorname{Spec}(R)$, we obtain exactly $m = \operatorname{spl}(R)$ equations that must be satified by elements of Λ. Conversely, if the b_j satisfy these equations, then $N :\cong V_1^{(b_1)} \oplus \cdots \oplus V_n^{(b_n)}$ has constant rank on each fiber of $\operatorname{Spec}(\widehat{R}) \longrightarrow \operatorname{Spec}(R)$. By Corollary 11.6, N is extended from an R-module, say $N \cong \widehat{F}$. Clearly $\widehat{F} \mid \widehat{M^{(u)}}$ if u is large enough, and it follows from Theorem 11.1 that $F \in +(M)$, whence $[b_1 \dots b_n]^{\mathrm{tr}} \in \Lambda$. □

In [Kat02] Karl Kattchee showed that, for each m, there is a finitely generated Krull monoid Λ that cannot be defined by m equations. Thus no single one-dimensional local ring can realize *every* finitely generated Krull monoid in the form $+(M)$ for a finitely generated module M.

two-dim-realization

§2 Realization theorem in dimension two

Suppose we have a finitely generated Krull semigroup H and a full embedding $H \subseteq \mathbb{N}_0^{(t)}$. By Theorem 11.11, we cannot realize this embedding in the form $+(M) \hookrightarrow +(\widehat{M})$ for a module M over a one-dimemsional ring unless H is actually an *expanded* subsemigroup of $\mathbb{N}_0^{(t)}$. If, however, we go to a two-dimensional ring, we can do so, though the ring that does the realizing is less tractable than the one-dimensional rings that realize expanded subsemigroups.

As in the last section, we need a criterion for an $\widehat{R}$-module to be extended from R. For general two-dimensional rings, we know of no such criterion, so we shall restrict to analytically normal domains. (A local domain $(R,\mathfrak{m})$ is *analytically normal* provided its completion $(\widehat{R},\widehat{m})$ is a normal domain.)

We recall from [Bou98, Chapter VII, Section 4.7] that over a Noetherian normal domain R one can assign to each finitely generated R-module M a
divisorclass *divisor class* $\mathrm{cl}(M) \in \mathrm{Cl}(R)$ in such a way that

1. Taking divisor classes is additive on exact sequences of finitely generated modules, and

2. if J is a fractional ideal of R, then $\mathrm{cl}(J)$ is the isomorphism class $[J^{**}]$ of the divisorial (= reflexive) ideal J^{**}.

The following result is Proposition 3 of [RWW99] (cf. also [Wes88, (1.5)]):

dimtwoextended **11.12 Proposition.** *Let* $(R,\mathfrak{m})$ *be a two-dimensional local ring whose* $\mathfrak{m}$*-adic completion* $(\widehat{R},\widehat{\mathfrak{m}})$ *is a normal domain. Let* N *be a finitely generated torsion-free* $\widehat{R}$*-module. Then* N *is extended from* R *if and only if* $\mathrm{cl}(N)$ *is in the image of the natural homomorphism* $\Phi : \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(\widehat{R})$.

Proof. Suppose $N \cong \widehat{R} \otimes_R M$. Then M is finitely generated and torsion-free, by faithfully flat descent. Choose a "Bourbaki sequence" (cf. [Bou98, Chapter VII, Sect. 4.9]

Bourbaki1 (11.12.1) $$0 \longrightarrow F \longrightarrow M \longrightarrow J \longrightarrow 0,$$

in which F is a free module and J is an ideal of R. Tensoring 11.12.1 with $\widehat{R}$, and using §2, we see that $\mathrm{cl}(N) = \mathrm{cl}(\widehat{R} \otimes_R J) = [(\widehat{R} \otimes_R J)^{**}] = [\widehat{R} \otimes_R J^{**}] = \Phi(\mathrm{cl}(J))$.

For the converse, choose a Bourbaki sequence

Bourbaki2 (11.12.2) $$0 \longrightarrow G \longrightarrow N \longrightarrow L \longrightarrow 0,$$

where G is $\widehat{R}$-free and L is an ideal of $\widehat{R}$. Then $\text{cl}(L) = \text{cl}(N)$, and since $\text{cl}(N)$ is in the image of Φ there is a divisorial ideal D of R such that $\widehat{R} \otimes_R D \cong L^{**}$. Then $V := L^{**}/L$ has finite length and hence is extended. By (2) of Lemma 11.5, L is extended. Of course G is extended; moreover, $\widehat{R}_P$ is a discrete valuation ring for each height-one prime ideal P, and it follows that $\text{Ext}^1_{\widehat{R}}(L, G)$ has finite length. Now (1) of Lemma 11.5 says that N is extended. □

As in the last section, we need to guarantee that the complete ring $\widehat{R}$ has a sufficiently rich supply of MCM modules.

bigclassgroup **11.13 Lemma** ([Wie01, Lemma 3.2])**.** *Let s be any positive integer. There is a complete normal domain B, containing $\mathbb{C}$, such that* $\dim(B) = 2$ *and* $\text{Cl}(B)$ *contains a copy of* $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Proof. Choose a positive integer d such that $(d-1)(d-2) \geqslant s$, and let V be a smooth projective plane curve of degree d over $\mathbb{C}$. Let A be the homogeneous coordinate ring of V for some embedding $V \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$. Then A is a two-dimensional normal domain, by [Har77, Chap. II, Exercise 8.4(b)]. By [Har77, Appendix B, Sect. 5], $\text{Pic}^0(V) \cong D := (\mathbb{R}/\mathbb{Z})^{2g}$, where $g = \frac{1}{2}(d-1)(d-2)$, the genus of V. Here $\text{Pic}^0(V)$ is the kernel of the degree map $\text{Pic}(V) \longrightarrow \mathbb{Z}$, so $\text{Cl}(V) = \text{Pic}(V) = D \oplus \mathbb{Z}\sigma$, where σ is the class of a divisor of degree 1. There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \text{Cl}(V) \longrightarrow \text{Cl}(A) \longrightarrow 0,$$

in which $1 \in \mathbb{Z}$ maps to the divisor class $\tau := [H \cdot V]$, where H is a line in $\mathbb{P}^2_{\mathbb{C}}$. (Cf. [Har77, Chap. II, Exercise 6.3].) Thus $\mathrm{Cl}(A) \cong \mathrm{Cl}(V)/\mathbb{Z}\tau$. Since τ has degree d, we see that $\tau - d\sigma \in D$. Choose an element $\delta \in D$ with $d\delta = \tau - d\sigma$. Recalling that $\mathrm{Cl}(V) = \mathrm{Pic}(V) = D \oplus \mathbb{Z}\sigma$, we define a surjection $f : \mathrm{Cl}(V) \longrightarrow D \oplus \mathbb{Z}/(d)$ by sending $x \in D$ to $(x, 0)$ and σ to $(-\delta, 1 + (d))$. Then $\ker(f) = \mathbb{Z}\tau$, so $\mathrm{Cl}(A) \cong D \oplus \mathbb{Z}/d\mathbb{Z}$.

Let $\mathfrak{P}$ be the irrelevant maximal ideal of A. By [Har77, Chap. II, Exercise 6.3(d)], $\mathrm{Cl}(A_{\mathfrak{P}}) \cong \mathrm{Cl}(A)$. The $\mathfrak{P}$-adic completion B of A is an integrally closed domain, by [ZS95, Chap. VIII, Sect. 13]. Moreover $\mathrm{Cl}(A_{\mathfrak{P}}) \longrightarrow \mathrm{Cl}(B)$ is injective by faithfully flat descent, so $\mathrm{Cl}(B)$ contains a copy of $D = (\mathbb{R}/\mathbb{Z})^{(d-1)(d-2)}$, which, in turn, contains a copy of $(\mathbb{R}/\mathbb{Z})^{(s)}$. □

We now have everything we need to prove our realization theorem for full subsemigroups of $\mathbb{N}_0^{(t)}$.

UFD **11.14 Theorem.** *Let t be a positive integer, and let H be a full subsemigroup of $\mathbb{N}_0^{(t)}$. Assume that H contains a strictly positive element h. Then there exist a two-dimensional local unique factorization domain R, a finitely generated reflexive (= MCM) R-module M, and a commutative diagram of semigroups*

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & \mathbb{N}_0^{(t)} \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ \mathrm{V}(+(M)) & \xrightarrow{\xi} & \mathrm{V}(+(\widehat{M})) \end{array},$$

where

1. *α and β are isomorphisms, and*
2. *$\alpha(h) = [M]$.*

Proof. Let G be the subgroup of $\mathbb{Z}^{(t)}$ generated by H, and write $\mathbb{Z}^{(t)}/G = C_1 \oplus \cdots \oplus C_s$, where each C_i is a cyclic group. Then $\mathbb{Z}^{(t)}/G$ can be embedded in $(\mathbb{R}/\mathbb{Z})^{(s)}$.

Let B be the complete local domain provided by Lemma 11.13. Since $\mathbb{Z}^{(t)}/G$ embeds in $\mathrm{Cl}(B)$, there is a homomorphism $\varphi : \mathbb{Z}^{(t)} \longrightarrow \mathrm{Cl}(B)$ with $\ker(\varphi) = G$. Let $\{e_1, \dots, e_t\}$ be the standard basis of $\mathbb{Z}^{(t)}$. For each $i \leqslant t$, write $\varphi(e_i) = [L_i]$, where L_i is a divisorial ideal of B representing the divisor class of $\varphi(e_i)$.

Next we use Heitmann's amazing theorem [Hei93], which implies that B is the completion of some local unique factorization domain R. For each element $m = (m_1, \dots, m_t) \in \mathbb{N}_0^{(t)}$, we let $\beta(m)$ be the isomorphism class of the B-module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. The divisor class of this module is $m_1[L_1] + \cdots + m_t[L_t] = \varphi(m_1, \dots, m_t)$. By Proposition 11.12, the module $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R-module if and only if its divisor class is trivial, that is, if and only if $m \in G \cap \mathbb{N}_0^{(t)}$. But $m \in G \cap \mathbb{N}_0^{(t)} = H$, since H is a full subsemigroup of $\mathbb{N}_0^{(t)}$. Therefore $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$ is the completion of an R-module if and only if $m \in H$. If $m \in H$, we let $\alpha(m)$ be the isomorphism class of a module whose completion is isomorphic to $L_1^{(m_1)} \oplus \cdots \oplus L_t^{(m_t)}$. In particular, choosing a module M such that $[M] = \alpha(h)$, we get the desired commutative diagram. □

§3 Exercises

11.15 Exercise. Complete the proof of Theorem 11.1.