MAT 631 — HOMEWORK 6

DUE ON TUESDAY 9 OCTOBER

1. Let $\mathbb{Q}(\sqrt{3})$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{3}$ (i.e. the intersection of all fields containing those things). Prove that $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a,\ b \in \mathbb{Q}\}$.

2. Define homomorphism of fields, and prove that every homomorphism of fields is injective.

3. Prove that the following are equivalent for an $n \times n$ matrix A over a field F:

(i) A is nonsingular, that is, has an inverse A^{-1}.

(ii) Left-multiplication by A is injective.

(iii) The columns of A are linearly independent.

(iv) The columns of A form a basis for F^n.

(v) For each $i = 1, \dots, n$, there exists a column vector b_i such that $Ab_i = e_i$ (standard basis vector).

(Hint: (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (i).) Apply this to determine the order of the group $\mathrm{GL}_n(\mathbb{F}_p)$ for all n, p.

4. Let B_1 and B_2 be bases for a finite-dimensional vector space V. Prove that there exist bases $C_0 = B_1,\ C_1,\ C_2,\ \dots,\ C_{n-1},\ C_n = B_2$ for V such that for each $i = 1, \dots, n-1$, we have $|C_i \cap C_{i+1}| = \dim V - 1$.