MAT 631 — HOMEWORK 7

DUE ON TUESDAY 16 OCTOBER

1. Let W_1 and W_2 be subspaces of the vector space V. The *sum* of W_1 and W_2 is the set

$$W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\}.$$

It is a subspace; in particular it is the smallest subspace containing W_1 and W_2. (You need not hand in a proof of either of the preceding assertions, but should check them for yourself.) Prove the following equality:

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

(You should address the case of infinite dimension, but it is relatively trivial. The main idea is to start with a finite basis of $W_1 \cap W_2$.)

2. Let P_n be the $\mathbb{R}$-vector space of polynomials $p(x) = a_0 + a_1 x + \cdots a_n x^n$ of degree at most n.

(a) Let D denote the (formal) derivative $\frac{d}{dx}$, considered as a linear operator $P_n \longrightarrow P_n$. Find $M_{\mathcal{B}}^{\mathcal{B}}(D)$, the matrix of D with respect to a convenient basis $\mathcal{B}$.

(b) Let $\lambda \in \mathbb{R}$ be arbitrary and prove that $1, x-\lambda, (x-\lambda)^2, \dots, (x-\lambda)^n$ is a basis for P_n.

3. Set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(a) Prove that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) Find the coordinate vector $[u]_{\mathcal{B}}$ of $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to $\mathcal{B}$.

(c) Let $\mathcal{C}$ be the standard basis for $\mathbb{R}^3$. Find an invertible matrix P such that $P[v]_{\mathcal{B}} = [v]_{\mathcal{C}}$ for all $v \in \mathbb{R}^3$. (There are 7s in denominators in this answer.)

(d) Define $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x,y,z) = (2x - z,\ y + 3z,\ x + y + z)$. Compute $M_{\mathcal{C}}^{\mathcal{C}}(T)$ and $M_{\mathcal{B}}^{\mathcal{B}}(T)$.