MAT 631 — HOMEWORK 3

DUE ON TUESDAY 17 SEPTEMBER 2013

1. (a) Assume that a group H acts on a set X. Define $x \sim y$ if there is some $h \in H$ such that $h \cdot x = y$. Prove that "$\sim$" is an equivalence relation (page 3 of D&F: symmetric, reflexive, transitive). The equivalence class containing x is called its *orbit*.

(b) Let G be a finite group, let H be a subgroup of G, and consider the action of H on G by left multiplication. In other words, an element $h \in H$ acts on an element $x \in G$ by $h \cdot x = hx$. Let $x \in G$ and let $Hx = \{hx \mid h \in H\}$ be its orbit. Prove that $|Hx| = |H|$, so that every orbit has the same size. (Set up a one-one correspondence between the elements of H and those of Hx.)

(c) Conclude *Lagrange's Theorem*: If G is a finite group and H is a subgroup, then $|H|$ divides $|G|$.

2. Let G be a group and X a subset of G. For an element $a \in G$, we write aXa^{-1} for the set of all elements of the form axa^{-1}, $x \in X$. The *normalizer* of X in G is

$$N_G(X) = \left\{a \in G \mid aXa^{-1} = X\right\}.$$

(a) Prove that $N_G(X)$ is a subgroup of G.

(b) Compute $N_{D_8}(\{1, \tau\})$.

(c) Compute $N_{S_3}(\{1, (1\ 2\ 3), (1\ 3\ 2)\})$.

3. Let G be a group and let H and K be subgroups of G.

(a) Prove that $H \cap K$ is a subgroup of G.

(b) Give an example to show that $H \cup K$ need not be a subgroup of G.

(c) Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup. (Do not assume that the collection is countable.)

(d) Let $x \in G$, and prove that $\langle x \rangle = \bigcap K$, where the intersection is over all subgroups K containing x.

4. Compute the lattice of subgroups of C_{75}.

5. Let $\varphi\colon G \longrightarrow H$ be a homomorphism. Prove that $\ker\varphi$ is a subgroup of G. Prove that φ is one-to-one if and only if $\ker\varphi$ is the trivial subgroup.