MAT 631 — HOMEWORK 5

DUE ON TUESDAY 2 OCTOBER 2013

1. Let $H \leq K \leq G$ be groups. Prove that $|G : H| = |G : K|\ |K : H|$. (Do not assume that G is finite nor that either subgroup is normal.)

2. Let G be a group and let H be a subgroup of index n.

(a) Prove that there is a homomorphism $\varphi\colon G \longrightarrow S_n$, with kernel a normal subgroup of H. (Hint: the action of G on left cosets of H by left-multiplication.)

(b) If p is the smallest *prime factor* of $|G|$, prove that H is normal in G. (Hint: apply 1^{st} Iso Thm and then Lagrange to φ above.)

3. Part (b) of this problem was on the August 2010 Prelim.

(a) Let A and B be subgroups of G such that $\gcd(|A|,\ |B|) = 1$. Prove that $A \cap B = \{1\}$.

(b) Let G, H, K be groups of orders 35, 60, and 42, respectively. Assume that there are homomorphisms $\varphi\colon G \longrightarrow H$ and $\psi\colon G \longrightarrow K$ such that $\ker\varphi \neq G$ and $\ker\psi \neq G$. Prove that $\ker\varphi \cap \ker\psi = \{1\}$.

4. (August 2011 Prelim) Let G be a group and let H, K be normal subgroups of G such that $H \cap K = \{1\}$. Prove that $HK \cong H \times K$. (The prelim version of this problem was misstated: both H and K need to be normal. Bonus problem: give a counterexample if K is not assumed to be normal in G.)