MAT 631 — HOMEWORK 6

DUE ON TUESDAY 9 OCTOBER 2013

1. Prove the j^{th} and k^{th} Isomorphism Theorems:

(a) If H and N are subgroups of G with N normal in G, then

$$\frac{H}{H \cap N} \cong \frac{HN}{N}.$$

(b) If $H < K < G$, both normal in G, then K/H is normal in G/H, and

$$\frac{G/H}{K/H} \cong G/K.$$

2. Let $\mathbb{Q}(\sqrt{2})$ be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\sqrt{2}$ (i.e. the intersection of all fields containing those things). Prove that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a,\ b \in \mathbb{Q}\}$.

3. Define homomorphism of fields, and prove that every non-zero homomorphism of fields is injective.

4. Let A be an $n \times n$ matrix over a field F.

(a) Prove that the following are equivalent:

(i) A is nonsingular, that is, has an inverse A^{-1}.

(ii) Left-multiplication by A is injective.

(iii) The columns of A are linearly independent.

(iv) The columns of A form a basis for F^n.

(v) For each $i = 1, \ldots, n$, there exists a column vector b_i such that $Ab_i = e_i$ (standard basis vector).

(Hint: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i) worked for me.)

(b) Apply part (a) to determine the order of the group $\mathrm{GL}_2(\mathbb{F}_p)$ for all p, using the fact that the span of a single vector in $(\mathbb{F}_p)^2$ has p elements in it. Bonus: conjecture a formula for $|\mathrm{GL}_n(\mathbb{F}_p)|$ for arbitrary n and (double bonus) prove it.