MAT 631 — HOMEWORK 10

DUE ON THURSDAY 7 NOVEMBER 2013

1. Prove that a group of any of the following orders is not simple.

(a) $459 = 3^3 \cdot 17$

(b) $2907 = 3^2 \cdot 17 \cdot 19$.

(c) $6545 = 5 \cdot 7 \cdot 11 \cdot 17$

(d) $300 = 2^2 \cdot 3 \cdot 5^2$

2. Let x and y be elements of a group, and define the *commutator* $[x,y] = x^{-1}y^{-1}xy$. Let G' be the subgroup generated by commutators. (Caution: this is not necessarily the *set of all* commutators, but is the smallest subgroup of G containing that set.) Prove that G/G' is the largest Abelian quotient of G, in the following sense: if N is a normal subgroup of G such that G/N is Abelian, then $G' \leq N$; and, if $G' \leq H \leq G$ then H is normal in G and G/H is Abelian.

3. Let G be a group of order pq, where $p < q$ are primes. Prove that $G = PQ$, where P and Q are Sylow subgroups. If in addition $q \not\equiv 1 \bmod p$, conclude that G is Abelian. (Hint: consider $[x,y]$, where $x \in P$ and $y \in Q$.)

4. Prove that in a group of order 105 both the Sylow 5- and 7-subgroups are normal. (Hint: This one is hardish. First prove that either one or the other is, then consider the subgroup PQ of order 35 where $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_7(G)$. Show that PQ is normal in G, and use the previous problem to show that it has a unique Sylow 5-subgroup, namely P. Conclude that P is unique. The same reasoning applies to Q.)