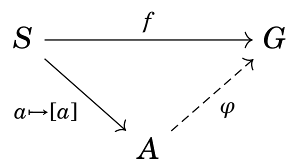
MAT 631 — HOMEWORK 11

DUE ON THURSDAY 14 NOVEMBER 2013

1. Let S be a set and $F = F_S$ the free group on S. Let F' be the commutator subgroup of F. Set $A = A_S = F/F'$, and call it the *free Abelian group* on S. Prove the universal mapping property of the free Abelian group: for any function $f : S \longrightarrow G$, where G is an Abelian group, there exists a unique group homomorphism $\varphi : A \longrightarrow G$ so that the diagram



commutes. Conclude that if $|S| = n < \infty$, then $A_S \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n factors).

2. Show that every square matrix over $\mathbb{R}$ is uniquely a sum of a symmetric matrix and a skew-symmetric matrix. Conclude that every bilinear form $\langle\ |\ \rangle$ on a real vector space V is uniquely a sum of a symmetric form and a skew-symmetric form.

3. Let $\langle\ |\ \rangle$ be a symmetric bilinear form on a vector space V over a field F. Define the *quadratic form* associated to $\langle\ |\ \rangle$ to be the function $q : V \longrightarrow F$ defined by $q(v) = \langle v|v\rangle$. Assume that $2 \neq 0$ in F (equivalently, $\frac{1}{2} \in F$) and prove that the bilinear form can be recovered from q.

4. Let $V = \mathbb{R}^n$ and $G = \mathrm{GL}_n(\mathbb{R})$. Define an action of G on the set of quadratic forms q by

$$(X \bullet q)(v) = q(X^T v)$$

for an invertible matrix X. Prove that if A is the symmetric matrix of the bilinear form corresponding to q, then the matrix of the bilinear form corresponding to $X \bullet q$ is XAX^T.