MAT 631 — HOMEWORK 12

DUE ON THURSDAY 21 NOVEMBER 2013

1. Let V be a finite-dimensional real vector space with a non-degenerate symmetric bilinear form $\langle - | - \rangle$. Let W be a subspace, and $\mathscr{B} = \{u_1, \dots, u_k\}$ an orthonormal basis for W.

(a) If $v \in W$ prove that the i^{th} entry of $[v]_{\mathscr{B}}$ is $\langle v \mid u_i \rangle$.

(b) For $v \in V$, define $\operatorname{proj}_W v = \sum_{i=1}^k \langle v \mid u_i \rangle\ u_i$, the *orthogonal projection* of v onto W. Prove that $v - \operatorname{proj}_W v \in W^\perp$.

(c) Prove that $V = W \oplus W^\perp$.

(d) Conclude that $W^{\perp\perp} = W$.

2. Keep the notation of the previous problem, except that $\langle - | - \rangle$ may be degenerate. Let $N = V^\perp$ be the nullspace. Prove that the quotient V/N has a well-defined and non-degenerate bilinear form induced from $\langle - | - \rangle$. Conclude that $W^{\perp\perp} = W + V^\perp$.

3. Let $u \in \mathbb{R}^d$ be a unit vector and set $W = \operatorname{span}(u)^\perp$ (with respect to the dot product). The *reflector across* W is $P_u = I_d - 2uu^T$.

(a) Prove that P_u is orthogonal.

(b) For a vector $v \in V$, show that there exist unique $c \in \mathbb{R}$ and $w \in W$ such that $v = cu + w$. (Hint: previous problem.)

(c) With the notation of the previous part, show that $P_u v = -cu + w$.

(d) Let x and y be two vectors in $\mathbb{R}^d$ with $|x| = |y|$. Find a unit vector u such that $P_u x = y$.

4. Let A be a hermitian matrix and $\lambda \in \mathbb{C}$ an eigenvalue for A. Prove that λ is real. (Hint: $(v^* A v)^*$.)