MAT 631 — HOMEWORK 13

DUE ON THURSDAY 5 DECEMBER 2013

1. Let $T \colon V \longrightarrow V$ be a linear operator on a hermitian space V, with adjoint operator $T^* \colon V \longrightarrow V$. Prove that $\ker T = (\operatorname{im} T^*)^\perp$. (Prelim problem from August '06, August '07, January '08, and August '13.)

2. Let A be a square complex matrix. Prove that the following conditions (for A to be hermitian) are equivalent.

(i) $A^*A = I$;

(ii) A preserves the standard hermitian product on $\mathbb{C}^n$;

(iii) A preserves length of vectors in $\mathbb{C}^n$.

(Prelim problem from January '07, January '13, and August '13.)

3. Prove that a square complex matrix A is hermitian iff z^*Az is real for all $z \in \mathbb{C}^n$. (Prelim problem from January '09 and August '12.)

4. Given a bilinear form $\langle - | - \rangle$ on a finite-dimensional vector space V over a field F, and a basis B for V, let A be the matrix of the form with respect to B. Define $\delta = \det A$.

(i) Give an example to show that δ is *not* well-defined, i.e. not independent of choice of the basis B.

(ii) Observe that $F^{\times 2} := \{\alpha^2 \mid \alpha \in F\}$ is a subgroup of the multiplicative group $F^\times$, and define the *discriminant* Δ of $\langle - | - \rangle$ by

$$\Delta = \begin{cases} 0 & \text{if } \delta = 0; \\ \text{the image of } \delta \text{ in } F^\times / F^{\times 2} & \text{otherwise.} \end{cases}$$

Prove that Δ is well-defined.

(iii) (bonus) Prove that $\mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ is infinite, and that $\mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} = \{\pm 1\}$ for all primes $p > 2$.

(I've never managed to get this one on a prelim, but some day....)