Pieri Maps and the Bound Young Quiver

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Background: Representations of $\mathrm{GL}(V)$

Convention

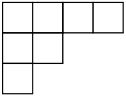
Throughout K is a field of characteristic 0 and V is a vector space of dimension $d < \infty$.

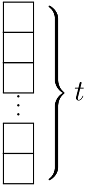
Recall

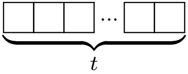
- The irreducible polynomial representations of $\mathrm{GL}(V)$ are in one-one correspondence with integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d)$ with $\lambda_i \geq 0$.
- The irreducible representation corresponding to λ is the Schur functor $L^\lambda V$.
- Every polynomial representation decomposes uniquely as a direct sum of $L^\lambda V$'s.

Background: Young diagrams

We identify the Schur functor $L^{\lambda}V$ with the partition λ, and identify both with the Young diagram of the partition:

$(4,\ 2,\ 1) \quad \longleftrightarrow$ 

$\underbrace{(1,\ 1,\ \ldots,\ 1)}_{t} \quad \longleftrightarrow$  $\quad \longleftrightarrow \quad \bigwedge^{t} V$

$(t) \quad \longleftrightarrow$  $\quad \longleftrightarrow \quad \mathrm{Sym}_t\, V$

Background: Pieri's Rules

Pieri's Rule (Basic Version)

$$V \otimes L^{\lambda} V = \bigoplus_{\lambda \nearrow \mu} L^{\mu} V \,,$$

where the sum is over all partitions μ with $\lambda \nearrow \mu$, that is, μ is obtained by adding a single box to λ.

Examples

$$V \otimes V = \square \otimes \square = \ydiagram{1,1} \oplus \ydiagram{2} = L^{(1,1)} V \oplus L^{(2)} V = \bigwedge^2 V \oplus \mathrm{Sym}_2\, V$$

$$V \otimes \bigwedge^2 V = \square \otimes \ydiagram{1,1} = \ydiagram{1,1,1} \oplus \ydiagram{2,1}$$

Background: Pieri's Rules

Pieri's Rule (Fancy Versions)

$$\bigwedge^t V \otimes L^\lambda V = \bigoplus_\mu L^\mu V \, ,$$

where the sum is over all partitions μ obtained by adding t boxes to λ in different rows.

$$\mathrm{Sym}_t V \otimes L^\lambda V = \bigoplus_\mu L^\mu V \, ,$$

where the sum is over all partitions μ obtained by adding t boxes to λ in different columns.

Background: Pieri's Rules

Pieri's Rule (Basic Version again)

$$V \otimes L^\lambda V = \bigoplus_{\lambda \nearrow \mu} L^\mu V\,.$$

Equivalently (by Maschke's theorem),

$$\mathrm{Hom}_{\mathrm{GL}(V)}(L^\mu,\ V \otimes L^\lambda V) = \begin{cases} K & \textit{if } \lambda \nearrow \mu, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

Any non-zero homomorphism $L^\mu V \longrightarrow V \otimes L^\lambda V$ is a split injection. Implicitly, Pieri's Rule amounts to a choice of a complete set of such retracts.

Pieri Systems

Definition

A Pieri system is a set of choices of non-zero $\mathrm{GL}(V)$-equivariant maps

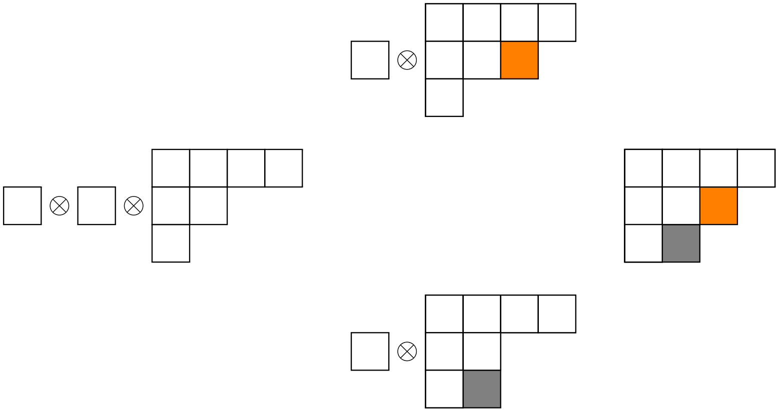
$$\chi_{\lambda,i}\colon \quad L^{\lambda+i}V \longrightarrow V \otimes L^{\lambda}V$$

for every partition λ and every i such that a box can be added to λ in row i.

- Each $\chi_{\lambda,i}$ is unique up to non-zero scalar multiples.
- $\chi_{\square,1}\colon \square\square \longrightarrow \square \otimes \square$ is $\mathrm{Sym}_2 V \longrightarrow V \otimes V$.
- $\chi_{\square,2}\colon \begin{smallmatrix}\square\\\square\end{smallmatrix} \longrightarrow \square \otimes \square$ is $\bigwedge^2 V \longrightarrow V \otimes V$.
- First example of a Pieri system constructed by P. Olver [1983ish, unpublished].

The Problem of Commutativity

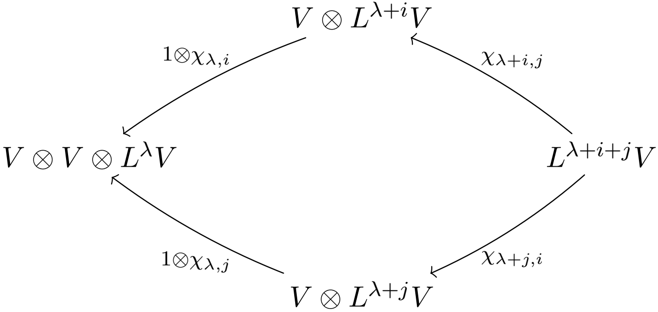
Consider adding 2 boxes to λ in rows i and j:



The Problem of Commutativity

Hope

We can choose the Pieri system $(\chi_{\lambda,i})_{\lambda,i}$ *in some coherent way so that all the squares*



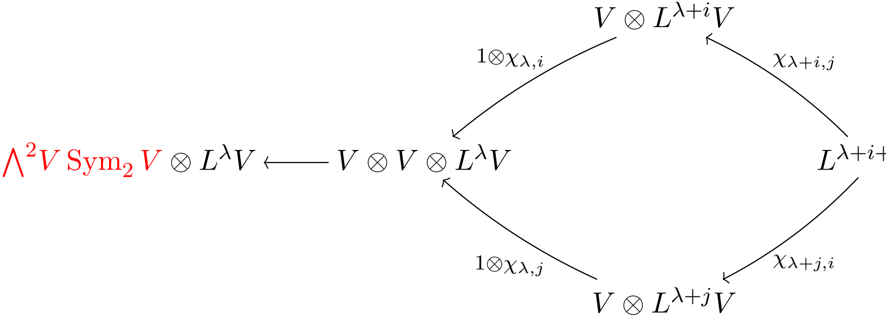
commute.

But $\mathrm{Hom}(L^{\lambda+i+j}V,\ V \otimes V \otimes L^{\lambda}V)$ is 2-dim'l; how to compare?

The Problem of Commutativity

$$V \otimes V = \bigwedge^2 V \oplus \mathrm{Sym}_2 V\,,$$

so compose with projection:



defines

$$\chi_{ij}^{-+},\ \chi_{ji}^{-+}\ \in \mathrm{Hom}(L^{\lambda+i+j}V,\ \bigwedge^2 V\, \mathrm{Sym}_2 V \otimes L^{\lambda} V) \quad \cong K.$$

Characteristic Ratios

Definition

The (exterior/symmetric) characteristic ratios of the Pieri system $(\chi_{\lambda,i})$ are the scalars

$$\gamma^{+}_{\lambda,i,j} = \frac{\chi^{+}_{ji}}{\chi^{+}_{ij}} \quad \text{and} \quad \gamma^{-}_{\lambda,i,j} = \frac{\chi^{-}_{ji}}{\chi^{-}_{ij}}\,.$$

Notice the squares all commute iff $\gamma^{+} = \gamma^{-} = 1$ for all λ, i, j.

Main Theorem

Theorem (Buchweitz-L-Van den Bergh)

Let $(\chi_{\lambda,i})$ be a Pieri system with characteristic ratios $(\gamma^{\pm}_{\lambda,i,j})$. Then for all λ, i, j,

$$\frac{\gamma^{+}_{\lambda,i,j}}{\gamma^{-}_{\lambda,i,j}} = \frac{u-1}{u+1}\,,$$

where

$$u = \frac{1}{(i-\lambda_i-1)-(j-\lambda_j-1)}\,.$$

- *In particular, the ratio of the characteristic ratios depends only on the added boxes $(i, \lambda_i + 1)$ and $(j, \lambda_j + 1)$.*
- *In particular, it's never the case that $\gamma^{+} = \gamma^{-} = 1$. So the squares never commute.*

Application to NCRs

Setup

- K a field of char. zero.
- X an $m \times n$ matrix of x_{ij}.
- $S = K[X]$
- $1 \leq t < \min(m, n)$
- $I_t(X)$ ideal of t-minors
- $R = S/I_t(X)$

Also G, F free S-modules of ranks n, m, so that

$$G \xrightarrow{X} F$$

is the generic homomorphism.
For a partition λ, set

$$N_\lambda = \mathrm{image}\left(\overline{L^\lambda F^*} \xrightarrow{\overline{L^\lambda X^T}} \overline{L^\lambda G^*}\right) .$$

Application to NCRs

$$N_\lambda = \text{image}\left(\overline{L^\lambda F^*} \xrightarrow{\overline{L^\lambda X^T}} \overline{L^\lambda G^*}\right)$$

Set

$$N = \bigoplus_{\lambda \in B_{t,m-t}} N_\lambda \,,$$

where the sum is over all λ with $\leq t$ rows and $\leq m-t$ columns.

Theorem (BLV)

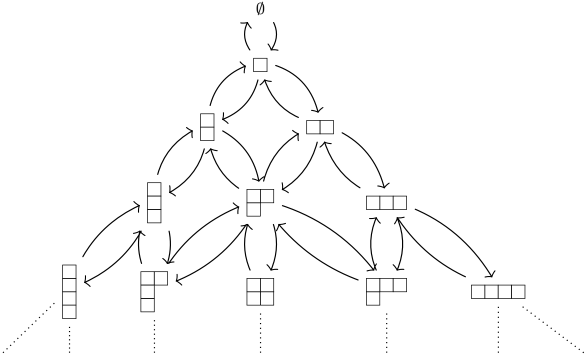
$A := \mathrm{End}_R(N)$ is a maximal Cohen-Macaulay R-module and has finite global dimension. ($\therefore$ it's a "non-commutative resolution" of R.) In particular each N_λ is MCM.

What does A look like?

Application to NCRs

Theorem (BLV)

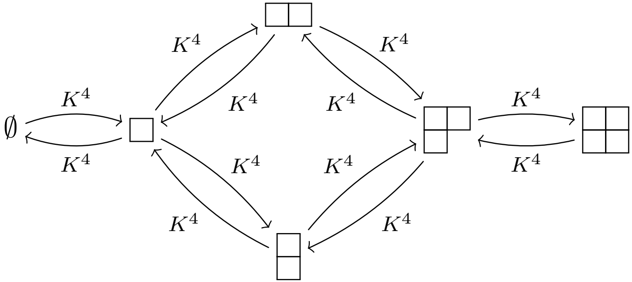
$A = \mathrm{End}_R(N)$ *is isomorphic to the path algebra of the truncated Young quiver*



subject to the relations induced by the Pieri maps (and their duals).

Example: $m = n = 4$, $t = 2$

Partitions fitting in a $t \times (m - t) = 2 \times 2$ box:



The path algebra of this quiver, with the Pieri relations, is a NC resolution of

$$R = K[x_{11}, \ldots, x_{44}]/I_2(X)\,.$$