MAT 534 — HOMEWORK 5

DUE ON FRIDAY 14 FEBRUARY

All these problems are from Chapter 6.

1. (#2) Determine all automorphisms of (the additive group) $\mathbb{Z}$. Determine the group structure of $\mathrm{Aut}(\mathbb{Z})$. Justify.

2. (#4,5) Show that $U(8) \cong U(12)$ but $U(8) \ncong U(10)$.

3. (#6) Prove that isomorphism is an equivalence relation. That is, for any groups G, H, and K, (i) $G \cong G$, (ii) if $G \cong H$ then $H \cong G$, and (iii) if $G \cong H$ and $H \cong K$ then $G \cong K$.

4. (#10) Let G be a group. Prove that the mapping $\varphi(g) = g^{-1}$ for all $g \in G$ is an automorphism if and only if G is Abelian.

5. (#28) The group $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}$ (under matrix multiplication) is isomorphic to what familiar group? Write down an isomorphism and prove it.

6. (#31,32) Prove properties 1 and 4 of Theorem 6.3. (These say that the inverse function of an isomorphism is also an isomorphism, and that the image of a subgroup is a subgroup.)

7. (#43) Let G be a group and let $g \in G$. If $z \in Z(G)$, show that the inner automorphism induced by g is the same as the inner automorphism induced by zg (that is, the mappings φ_g and φ_{zg} are equal).

Also suggested, but not to hand in:

(A) (#7) Prove that S_4 is not isomorphic to D_{12}.

(B) (#14) Determine all automorphisms of $\mathbb{Z}_6$. Determine the group structure of $\mathbb{Z}_6$.

(C) (#15) If G is a group, prove that $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are groups.

(D) (#27) Let $r \in U(n)$. Prove that the mapping $\varphi\colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ defined by $\varphi(k) = rk \bmod n$ is an automorphism of $\mathbb{Z}_n$.

(E) (#42) Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping $g \mapsto g^2$ is an automorphism of G. Show, by example, that there is an infinite Abelian group for which the mapping $g \mapsto g^2$ is one-to-one and operation-preserving but not an automorphism.

(F) (#45) Suppose that g and h induce the same inner automorphism of a group G. Prove that $h^{-1}g \in Z(G)$.