MAT 733 — HOMEWORK 3

DUE ON WEDNESDAY 5 MARCH

All rings are commutative with 1.

1. Let I be an ideal of a ring R. A prime ideal $\mathfrak{p}$ containing I is said to be *minimal* over I if there is no prime ideal $\mathfrak{q}$ with $I \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$. Suppose R is Noetherian. Prove that every prime minimal over I is an associated prime of R/I. Conclude that there are only finitely many primes minimal over I. (Hint: localize at the minimal prime.)

2. Let $R \subset S$ be rings, and $u \in S$. Prove that the following are equivalent:

(a) u is *integral* over R, that is, satisfies a monic polynomial equation with coefficients in R.

(b) $R[u]$ is a finitely generated R-module.

(c) $R[u]$ is contained in a subring B of S such that B is a finitely generated R-module.

(d) There exists a *faithful* $R[u]$-module M which is finitely generated over R.

Recall that M is *faithful* if the only element of the ring that annihilates M is 0. You will want to use the *determinant trick,* which we will prove in lecture. Remind me if I forget.

3. Let I be an ideal of a ring R. For each $x \in R$ define $o(x) = n$ if $x \in I^n \setminus I^{n+1}$, and $o(x) = \infty$ if $x \in \bigcap_{n\geq 0} I^n$. Now define a real-valued function ρ on $R \times R$ by $\rho(x, y) = 2^{-o(x-y)}$. (Interpret $2^{-\infty}$ as 0.) Show that if R is Noetherian and I is contained in the Jacobson radical then ρ defines a metric on R. (In fact it is an *ultrametric*, whatever that means.) The resulting topology is called the I-adic topology on R.

4. It's easy to construct a ring of infinite Krull dimension: $S = k[x_1, x_2, \ldots]$ will do. This problem follows Nagata's 1958 construction of a *Noetherian* ring of infinite dimension. Let k be a field, set $S = k[x_1, x_2, \ldots]$, and let A be an arbitrary commutative ring.

(a) For $J \subseteq I$ ideals of A, show that $J = I$ if and only if $(I/J)_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ of A.

(b) Assume that $A_{\mathfrak{m}}$ is Noetherian for every maximal ideal $\mathfrak{m}$, and that every non-zero $f \in A$ is contained in only finitely many maximal $\mathfrak{m}$. Prove that A is Noetherian. (Hint: to show a given ideal I is finitely generated, pick some $0 \neq f \in I$, and lift all the generators of $I_{\mathfrak{m}_i}$, as $\mathfrak{m}_i$ runs over the maximal ideals containing f, to A. Together with f they generate I by (a).)

(c) Prove that the ideal $(x_1, \ldots, x_n)S$ has height n for every n.

(d) For each $i \geq 0$, let $\mathfrak{p}_i = (x_{2^i}, \ldots, x_{2^{i+1}-1})S$. Let U be the complement in S of the union of the p_i, and set $R = U^{-1}S$. Prove that R is Noetherian and that $p_i R$ has height 2^i, so that $\dim R = \infty$.

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5. Let R be a Noetherian ring and $\mathfrak{p} \in \operatorname{Spec} R$.

(a) Prove a converse to the Krull Principal Ideal Theorem: if $\mathfrak{p}$ has height n, then $\mathfrak{p}$ is minimal over an n-generated ideal. (Hint: avoid the minimal primes of $(a_1, \ldots a_{n-1})$.

(b) Conclude that

$$\dim R = \inf\{n \mid \exists \text{ maximal ideal } \mathfrak{m} \text{ which is minimal over an } n\text{-generated ideal}\}.$$

(c) Conclude that $\dim R[x] = \dim R + 1$.