MAT 733 — HOMEWORK 5

DUE ON WEDNESDAY 16 APRIL

All rings are commutative with 1.

1. (E. Noether) Let $k \subseteq A$ be a ring extension where k is a field. Assume $a_1, \ldots, a_n \in A$ and set $S = k[a_1, \ldots, a_n]$. Let G be a finite group of ring automorphisms of S which fix k. Let T be the subring of S consisting of all elements fixed by all the automorphisms in G. Prove that T is a finitely generated k-algebra, and is in particular Noetherian.

(Hints: let x be an indeterminate over k and for $i = 1, \ldots, n$ let

$$\begin{aligned} f_i(x) &= \prod_{\sigma \in G} x - \sigma(a_i) \\ &= x^m + p_{i1}x^{m-1} + \cdots + p_{im} \end{aligned}$$

where $m = |G|$. Set $R = k[p_{11}, \ldots, p_{nm}]$. Then R is Noetherian, $R \subseteq T \subseteq S$, and S is integral over R (show these things). Conclude that T is a finitely generated R-module.)

2. Let R be a Noetherian ring, M a finitely generated R-module, and N an arbitrary R-module. Let S be a flat R-algebra. Prove that

$$\operatorname{Hom}_R(M, N) \otimes_R S = \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S).$$

(Hint: Define an S-linear map from left to right by $f \otimes s \mapsto s \cdot (f \otimes 1_S))$. Prove it is an isomorphism for M free of finite rank, and then apply both sides (as functors in M) to an exact sequence $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$.)

3. Let R, M, N be as above, and $\mathfrak{p} \in \operatorname{Spec} R$. Prove that

$$\operatorname{Hom}_R(M, N)_\mathfrak{p} = \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}).$$

4. Let $(R, \mathfrak{m})$ be a one-dimensional Noetherian local ring and assume $\mathfrak{m} = (x)$ is a principal ideal. Prove that R is a domain (and hence a DVR). (Hint: KIT.)

5. Let R be a domain. Prove that R is a valuation ring if and only if the ideals of R are linearly ordered: for every pair of ideals I, J, either $I \subseteq J$ or $J \subseteq I$.

6. (Bonus: There is part of this I don't know how to do.) Let $S = k[x, y]$, where k is a field.

(a) Put the lexicographic ordering on $\mathbb{R} \times \mathbb{R}$, that is, $(r, s) > (t, u)$ means either $r > s$ or $r = s$ and $t > u$. Define $v_1 \colon S \longrightarrow (\mathbb{R} \times \mathbb{R}) \cup \{\infty\}$ first on monomials by $v_1(x^i y^j) = (i, j)$; extend to arbitrary $0 \neq f \in S$ by letting $v_1(f)$ be the minimum value of v_1 on its monomials, and set $v_1(0) = \infty$. Observe that v_1 is multiplicative, so extends to a function $v_1 \colon k(x, y) \longrightarrow (\mathbb{R} \times \mathbb{R}) \cup \{\infty\}$. Prove that v_1 is a valuation, and determine the valuation group and valuation ring.

(b) Put the usual ordering on $\mathbb{R}$. For a nonzero polynomial $f \in S$, write $f = \sum_{i,j} \alpha_{ij} x^i y^j$, and define $v_2 \colon S \longrightarrow \mathbb{R} \cup \{\infty\}$ by $v_2(f) = \min\{i + j\sqrt{2}\}$ for $f \neq 0$ and $v_2(0) = \infty$. Extend v_2 to $k(x, y)$ as above, prove that v_2 is a valuation, and determine the valuation group and valuation ring.