MAT 733 — HOMEWORK 6

DUE ON MONDAY 28 APRIL

This is a long one. **Choose 5 problems to do.**

1. Let $(R,\mathfrak{m})$ be a Noetherian local ring. Prove that the following are equivalent for a finitely generated R-module M:

(a) M has finite length, that is, a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} = k$ for each i.

(b) $M_\mathfrak{p} = 0$ for every non-maximal prime $\mathfrak{p} \in \operatorname{Spec} R$.

(c) $\operatorname{Ann}_R M$ is $\mathfrak{m}$-primary.

(d) $\operatorname{Ass}_R M = \{\mathfrak{m}\}$.

2. Let R be a Noetherian ring and M a finitely generated R-module.

(a) Prove that M has a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each successive quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Supp} M$.

(b) Conclude that if $\operatorname{Ext}^i_R(R/\mathfrak{p}, N) = 0$ for all $\mathfrak{p} \in \operatorname{Supp} M$, then $\operatorname{Ext}^i_R(M,N) = 0$. (The same proof works for Tor.)

(c) Conclude that if $(R,\mathfrak{m},k)$ is local and M is any R-module of finite length, then

$$\operatorname{depth} N = \min\left\{ n \;\middle|\; \operatorname{Ext}^n_R(M,N) \neq 0 \right\}.$$

3. Prove that $H_1(x,0;R) \cong H_0(x;R) \oplus H_1(x;R)$.

4. Let R be a Noetherian ring and $x_1,\ldots,x_n$ an R-sequence. Prove that $\operatorname{height}(x_1,\ldots,x_n) = n$. (If I is an arbitrary ideal, we define $\operatorname{height} I := \min\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \in V(I)\}$.)

5. Let R be a Noetherian ring and I a proper ideal. The *grade* of I is $\operatorname{depth}_I(R)$. Prove that $\operatorname{grade} I \le \operatorname{height} I$ and $\operatorname{grade} I \le \operatorname{pd}_R(R/I)$.

6. The Depth Lemma says that if R is a Noetherian ring, I is an ideal of R, and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence of finitely generated R-modules with $IA \neq A$, $IB \neq B$, and $IC \neq C$, then

$$\begin{aligned}
\operatorname{depth}_I A &\ge \min\left\{\operatorname{depth}_I B, \operatorname{depth}_I C + 1\right\}\\
\operatorname{depth}_I B &\ge \min\left\{\operatorname{depth}_I A, \operatorname{depth}_I C\right\}\\
\operatorname{depth}_I C &\ge \min\left\{\operatorname{depth}_I A - 1, \operatorname{depth}_I B\right\}.
\end{aligned}$$

(a) Pick one of these inequalities to prove.

(b) Conclude that if $\operatorname{depth}_I B > \operatorname{depth}_I C$, then $\operatorname{depth}_I A = \operatorname{depth}_I C + 1$. (You can use any of the inequalities, not just the one you proved.)

7. Let R and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be as above. Let $x \in R$ be a nonzerodivisor on A and C. Prove x is a nonzerodivisor on B.