

ADDENDUM TO “ENDOMORPHISM RINGS OF FINITE GLOBAL DIMENSION”

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ABSTRACT. Example 12 of [Leu07] ends with the sentence “I do not know whether the endomorphism ring $\text{End}_R(R \oplus \omega)$ has finite global dimension.” It does, and the global dimension is 4, one more than $\dim R$. The proof is elementary: it uses only the modules of covariants of a one-dimensional torus acting on a polynomial ring in four variables, Koszul complexes, and Hilbert–Burch resolutions.

1. THE QUESTION

Let k be a field and let $R = k[[x, y, z, u, v]]/I$, where I is generated by the 2×2 minors of $\begin{bmatrix} x & y & u \\ y & z & v \end{bmatrix}$; this is the complete scroll of type $(2, 1)$ [Yos90, (16.13)], [LW12, §16.1]. In [Leu07, Example 12] the nonfree indecomposable maximal Cohen–Macaulay R -modules are listed as the canonical module $\omega \cong (u, v)R$, $M = \text{syzy}_1^R(\omega) \cong (x, y, u)R$, $N = \text{syzy}_2^R(\omega)$ of rank two, and $L = \text{Hom}_R(M, \omega) \cong (x, y, z)R$. The endomorphism rings $\text{End}_R(R \oplus \omega)$ and $\text{End}_R(R \oplus M)$ are the only ones of the form $\text{End}_R(D)$, with D a nonfree MCM module, that are themselves MCM.

Theorem 1. *Let $\Lambda = \text{End}_R(R \oplus \omega)$.*

- (1) $\Lambda \cong \text{End}_R(R \oplus M)$ as rings.
- (2) $\text{gldim } \Lambda = 4$. *Of the two simple Λ -modules, the one at the vertex R has projective dimension 3 and the one at the vertex ω has projective dimension 4.*

Nothing below uses the characteristic of k .

Corollary 2. *No ring $\text{End}_R(D)$, with D a nonfree MCM R -module, satisfies [Leu07, Definition 10]: the two that are MCM have global dimension $4 \neq \dim R$.*

2. THE RING OF INVARIANTS AND ITS MODULES OF COVARIANTS

Set $B = k[s, t, a, b]$, graded by $\deg s = \deg t = 1$, $\deg a = -2$, $\deg b = -1$, and write B_n for the degree- n part of B . Put $R_0 = B_0$. This is the situation of [Yos90, Chapter 16] with the weights $A = {}^t(2, 1, -1, -1)$: Yoshino’s grading is ours negated, with x_1, x_2, x_3, x_4 corresponding to a, b, s, t , and his S_c is our B_{-c} . He works over an algebraically closed field of characteristic 0, but the statements we use, and their proofs, are elementary and do not use this.

Proposition 3.

- (i) *The ring R_0 is generated by $x = s^2a$, $y = sta$, $z = t^2a$, $u = sb$, and $v = tb$, and the natural map $k[x, y, z, u, v]/I \rightarrow R_0$ is an isomorphism. It is a domain of dimension three, and R is its completion at the irrelevant ideal.*
- (ii) *The R_0 -module B_n is generated by the monomials of degree n in s, t when $n \geq 1$, and $B_{-1} = (b, sa, ta)$.*

- (iii) For $i \in \{-1, 0, 1\}$ and all $j \in \mathbb{Z}$, $\text{Hom}_{R_0}(B_i, B_j) = B_{j-i}$, acting by multiplication.
 (iv) Multiplication by b , s and a gives isomorphisms

$$B_1 \cong (u, v), \quad B_{-1} \cong (x, y, u), \quad B_2 \cong (x, y, z)$$

of R_0 -modules.

Proof. (i) is [Yos90, (16.12), (16.13)]; see also [LW12, §16.1]. (iii) is [Yos90, (16.7)], whose hypothesis that $B_i B$ has height at least 2 holds because $B_1 B = (s, t)$, $B_{-1} B = (b, sa, ta)$ and $B_0 B = B$. The cases $n = 1, 2$ and $n = -1$ of (ii) occur in the proof of [Yos90, (16.12)], where it is also observed that B_1 is the canonical module. For completeness: a monomial of degree $n \geq 1$ has at least n factors s or t , and removing n of them leaves a monomial of degree 0. A monomial of degree -1 either contains b , and removing it leaves a monomial of degree 0, or contains no b and then contains a and at least one s or t , and removing sa or ta leaves a monomial of degree 0. (iv) By (ii), $bB_1 = (sb, tb) = (u, v)$, $sB_{-1} = (sb, s^2a, sta) = (u, x, y)$ and $aB_2 = (x, y, z)$, and multiplication by a nonzero element of the domain B is injective. $\square$

3. THE ALGEBRA Λ AND ITS PROJECTIVE MODULES

We use right modules; the global dimension of Λ is the same on both sides [Ram69, Cor. 1.3]. All the R_0 -modules and all the exact sequences below remain exact after completion at the irrelevant ideal, and Hom between finitely generated modules commutes with completion. We therefore keep the notation B_n for the completed modules and R for R_0 completed, with maximal ideal $\mathfrak{m} = (x, y, z, u, v)$. By Proposition 3(i),(iv) and [Leu07, Example 12], $B_0 = R$, $B_1 \cong \omega$ and $B_{-1} \cong M$.

Let $N = B_0 \oplus B_1$, $\Lambda = \text{End}_R(N)$ and let e_0, e_1 be the projectors $N \twoheadrightarrow B_i \hookrightarrow N$. By Proposition 3(iii), $e_j \Lambda e_i = \text{Hom}_R(B_i, B_j) = B_{j-i}$, with composition given by multiplication; that is,

$$\Lambda = \begin{bmatrix} B_0 & B_{-1} \\ B_1 & B_0 \end{bmatrix}.$$

The indecomposable projective right modules are $P_0 = e_0 \Lambda$ and $P_1 = e_1 \Lambda$, and $P_i \rightarrow P_j$ is left multiplication by an element of $e_j \Lambda e_i = B_{j-i}$. For $m \in \mathbb{Z}$ put

$$Q_m = \text{Hom}_R(N, B_m), \quad \text{so that} \quad Q_m e_0 = B_m, \quad Q_m e_1 = B_{m-1},$$

and $P_0 = Q_0$, $P_1 = Q_1$. We write a Λ -module X as the pair (Xe_0, Xe_1) . Elements of B_1 (such as s, t) act from the second entry to the first, and elements of B_{-1} (such as b, sa, ta) from the first to the second.

Lemma 4. Let $J = \begin{bmatrix} \mathfrak{m} & B_{-1} \\ B_1 & \mathfrak{m} \end{bmatrix}$. Then $J = \text{rad } \Lambda$ and $\Lambda/J \cong k \times k$. The simple right modules are $S_R = P_0/P_0 J$ and $S_\omega = P_1/P_1 J$, and $\text{rad } P_0 = (\mathfrak{m}, B_{-1})$, $\text{rad } P_1 = (B_1, \mathfrak{m})$. A homomorphism $P_i \rightarrow P_j$ has image in $\text{rad } P_j$ unless $i = j$ and it is an isomorphism.

Proof. By Proposition 3(ii), $B_{-1} B_1 = B_1 B_{-1} = (b, sa, ta)(s, t) = \mathfrak{m}$. So J is a two-sided ideal, $J^2 \subseteq \mathfrak{m} \Lambda$, and $\Lambda/J \cong k \times k$. Since Λ is module-finite over the complete local ring R , we have $\mathfrak{m} \Lambda \subseteq \text{rad } \Lambda$, hence $J \subseteq \text{rad } \Lambda$; as Λ/J is semisimple, $J = \text{rad } \Lambda$. The description of the radicals of $P_i = e_i \Lambda$ is $e_i J$. A map $P_i \rightarrow P_j$ is multiplication by some $\lambda \in B_{j-i}$; if $i \neq j$ then $\lambda \in J$, and if $i = j$ then $\lambda \in R$ lies in J unless it is a unit. $\square$

Lemma 5.

- (i) For each $m \geq 1$ there is an exact sequence $0 \rightarrow P_0^{m-1} \rightarrow P_1^m \rightarrow Q_m \rightarrow 0$. In particular $\text{pd } Q_m \leq 1$ for $m \geq 1$.
- (ii) There is an exact sequence $0 \rightarrow Q_3 \rightarrow Q_2 \oplus P_1 \rightarrow P_0 \rightarrow S_R \rightarrow 0$.
- (iii) For each $m \leq -1$ there is an exact sequence $0 \rightarrow Q_{m+3} \rightarrow Q_{m+2} \oplus Q_{m+1} \rightarrow Q_m \rightarrow 0$.

Proof.

(i) Let $\sigma_m : B_1^m \rightarrow B_m$ send (f_i) to $\sum_{i=1}^m s^{m-i} t^{i-1} f_i$. It is onto because B_m is generated by the monomials of degree m in s, t , each of which is one of these monomials times s or t . By the Hilbert–Burch theorem, the syzygies of $(s^{m-1}, s^{m-2}t, \dots, t^{m-1})$ over B are freely generated by $te_j - se_{j+1}$ for $1 \leq j \leq m-1$. Since all f_i have degree 1, the coefficients of the syzygy have degree 0, so the kernel of σ_m is B_0^{m-1} , embedded by $(r_j) \mapsto (\dots, tr_j - sr_{j-1}, \dots)$. The same argument with B_0 and B_{m-1} in place of B_1 and B_m gives the exact sequence $0 \rightarrow B_{-1}^{m-1} \rightarrow B_0^m \rightarrow B_{m-1} \rightarrow 0$. Together, these are the two entries of an exact sequence of Λ -modules $0 \rightarrow Q_0^{m-1} \rightarrow Q_1^m \rightarrow Q_m \rightarrow 0$.

(iii) and (ii) The Koszul complex of the regular sequence a, b on B is an exact sequence of graded modules

$$0 \rightarrow B(3) \rightarrow B(2) \oplus B(1) \rightarrow B \rightarrow k[s, t] \rightarrow 0,$$

where $B(d)_n = B_{n+d}$, the first map is $f \mapsto (-bf, af)$ and the second is $(g, h) \mapsto ag + bh$. In degree n it reads $0 \rightarrow B_{n+3} \rightarrow B_{n+2} \oplus B_{n+1} \rightarrow B_n \rightarrow k[s, t]_n \rightarrow 0$, and $k[s, t]_n = 0$ for $n < 0$. Taking the degrees $n = m$ and $n = m-1$ as the two entries of a Λ -module gives

$$0 \rightarrow Q_{m+3} \rightarrow Q_{m+2} \oplus Q_{m+1} \rightarrow Q_m \rightarrow (k[s, t]_m, k[s, t]_{m-1}) \rightarrow 0.$$

For $m \leq -1$ the last term is 0. For $m = 0$ it is $(k, 0)$, which is the simple module S_R . $\square$

4. PROOF OF THE THEOREM

(1) By Proposition 3(iii), $\text{End}_R(B_{-1} \oplus B_0)$ and $\text{End}_R(B_0 \oplus B_1)$ both have B_{j-i} in position (i, j) , with multiplication in B , so $i \mapsto i+1$ induces a ring isomorphism. Since $B_1 \cong \omega$ and $B_{-1} \cong M$, this is $\Lambda \cong \text{End}_R(R \oplus M)$.

(2) The simple module S_ω has projective dimension 4. The module $\text{rad } P_1 = (B_1, \mathfrak{m})$ is the image of $\pi : P_0^2 \rightarrow P_1$, $(p_1, p_2) \mapsto sp_1 + tp_2$: its entries are $sB_0 + tB_0 = B_1$ and $sB_{-1} + tB_{-1} = (u, x, y, v, y, z) = \mathfrak{m}$. The kernel of π consists of the pairs $(tq, -sq)$ with $q \in B_{-1}$ in the first entry and $q \in B_{-2}$ in the second, since s, t have no common factor in B ; so $\ker \pi \cong Q_{-1}$. This kernel lies in $\text{rad}(P_0^2) = (\mathfrak{m}, B_{-1})^2$, because $B_{-1}B_1 \subseteq \mathfrak{m}$, so π is a projective cover. By Lemma 5(iii) with $m = -1$ there is an exact sequence $0 \rightarrow Q_2 \rightarrow P_1 \oplus P_0 \rightarrow Q_{-1} \rightarrow 0$. Its first map is given by $f \mapsto (-bf, af)$, and $bB_2 \subseteq B_1$, $bB_1 \subseteq \mathfrak{m}$, $aB_2 \subseteq \mathfrak{m}$, $aB_1 \subseteq B_{-1}$ show that its image lies in $\text{rad}(P_1 \oplus P_0)$; hence $P_1 \oplus P_0 \rightarrow Q_{-1}$ is a projective cover. By Lemma 5(i) with $m = 2$ there is an exact sequence $0 \rightarrow P_0 \rightarrow P_1^2 \rightarrow Q_2 \rightarrow 0$, whose first map is multiplication by $(t, -s)$ and therefore lies in the radical by Lemma 4. So we have found the minimal projective resolution

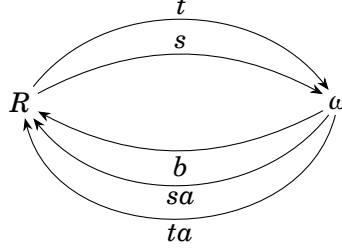
$$0 \rightarrow P_0 \rightarrow P_1^2 \rightarrow P_1 \oplus P_0 \rightarrow P_0^2 \rightarrow P_1 \rightarrow S_\omega \rightarrow 0,$$

and $\text{pd } S_\omega = 4$.

The simple module S_R has projective dimension 3. By Lemma 5(ii) there is an exact sequence $0 \rightarrow Q_3 \rightarrow Q_2 \oplus P_1 \rightarrow P_0 \rightarrow S_R \rightarrow 0$. Let $K = \ker(P_0 \rightarrow S_R)$. Since $\text{pd } Q_2, \text{pd } Q_3 \leq 1$ by Lemma 5(i), the sequence $0 \rightarrow Q_3 \rightarrow Q_2 \oplus P_1 \rightarrow K \rightarrow 0$ gives $\text{pd } K \leq 2$, so $\text{pd } S_R \leq 3$. For the reverse inequality, the functor $X \mapsto Xe_0$ from Λ -modules to R -modules is exact, and $P_0e_0 = R$, $P_1e_0 = B_1 \cong \omega$, $S_Re_0 = k$. Applying it to a projective resolution of S_R of length ℓ gives an exact sequence $0 \rightarrow X_\ell \rightarrow \dots \rightarrow X_0 \rightarrow k \rightarrow 0$ in which every X_i is a direct sum of copies of R and ω , hence has depth $3 = \dim R$. By the depth lemma, $\ell \geq 3$.

Finally, Λ is module-finite over the complete local ring R and $\Lambda/\text{rad } \Lambda = S_R \oplus S_\omega$, so $\text{gldim } \Lambda = \text{pd}(\Lambda/\text{rad } \Lambda) = \max\{3, 4\} = 4$ by [Ram69, Cor. 1.3]. $\square$

Remark 6. Since $B_{-1}B_1 = B_1B_{-1} = \mathfrak{m}$, the quotient $\text{rad } \Lambda/\text{rad}^2 \Lambda$ vanishes on the diagonal. So the quiver of Λ has no loops, and its arrows between the two vertices correspond to minimal generators of $\text{Hom}(R, \omega) = B_1$, namely s, t , and of $\text{Hom}(\omega, R) = B_{-1}$, namely b, sa, ta : two arrows $R \rightarrow \omega$ and three arrows $\omega \rightarrow R$.



Remark 7. In [Leu07, Example 11] the analogous ring $\text{End}_R(R \oplus \omega)$ is a twisted group ring and has global dimension $3 = \dim R$. Here Λ is an MCM R -module of finite global dimension, but its global dimension exceeds $\dim R$, so Λ is a non-commutative resolution that is not crepant in the sense of [Leu07, Definition 10].

Remark 8. Geometrically, $\text{Spec } R_0$ has a small resolution $X \rightarrow \text{Spec } R_0$, where X is the total space of $\mathcal{O}(-2) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$, and $\mathcal{O} \oplus \mathcal{O}(1)$ is a tilting bundle on X with endomorphism ring $\text{End}_{R_0}(R_0 \oplus B_1)$. So Λ is derived equivalent to X . We do not use this.

Proof of Corollary 2. By [Leu07, Example 12] the MCM endomorphism rings are $\text{End}_R(R \oplus \omega)$ and $\text{End}_R(R \oplus M)$, which are isomorphic by Theorem 1(1) and have global dimension 4 by Theorem 1(2). $\square$

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