AN INTRODUCTION TO MAXIMAL COHEN-MACAULAY MODULES

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Abstract. The first lecture is an overview of the theory of MCM modules (over CM local rings), from the definitions through to basic background results and maybe a couple of standard theorems. Probably also should mention the case of hypersurfaces.

The second lecture will focus on one or two examples to show that the subject is still an interesting field of research. Maybe a bit from the Sanya lectures.

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Many thanks to the organizers for the honor of giving this series of 2 lectures on an introduction to MCM modules, a topic close to my heart.

Part 1. The basics of MCM modules

Throughout, we work over a commutative Noetherian local ring $(R, \mathfrak{m})$ of (Krull) dimension d. Let's define the object of interest.

Definition. A finitely generated non-zero R-module M is maximal Cohen-Macaulay (MCM) if

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \qquad \text{for } i = 0, \dots, d-1.$$

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This means that the depth of M (first non-vanishing Ext) is d, the maximal possible value.

The MCM condition is a kind of higher torsion-freeness. It's hard to get much intuition from this definition. Notice that $i = 0$ says that the socle of M is zero, at least. Here is an equivalent definition (we'll prove the equivalence later): suppose R contains a field k and is either $\mathfrak{m}$-adically complete or is of finite type over k, so that there exists a polynomial or power series ring $T \subseteq R$ over which R is module-finite. Then M is MCM over R if and only if it is free over T.

As a special case, the ring R itself is Cohen-Macaulay (CM) if it is MCM as a module. In other words (in the nice cases mentioned before) R is a finitely generated free module over a polynomial/power series subring.

This is not an unreasonable restriction; many natural classes of rings are CM.

- All Artinian (0-dimensional) local rings are CM, in particular finite-dimensional algebras over a field.
- A one-dimensional domain (e.g. the coordinate ring of an irreducible curve singularity) is CM.
- A two-dimensional normal (=integrally closed) domain is CM.
- Any regular local ring (e.g., coordinate ring of a smooth point on an algebraic variety) is CM. More generally, a complete intersection ring is CM.
- Rings of invariants of group actions, both for finite groups and linearly reductive groups, are CM (Hochster-Roberts 1974).

So we are concerned with MCM modules over CM local rings. What do they look like in low dimensions?

- If $d = 0$ then every finitely generated module is MCM.
- If R is a one-dimensional domain then M is MCM iff it is torsion-free, i.e. every nonzero element of R is a nonzerodivisor on M.

- If R is a two-dimensional normal domain then M is MCM iff it is reflexive, that is, the natural map $M \longrightarrow M^{**}$ (where $-^* = \operatorname{Hom}_R(-, R)$) is an isomorphism.

And over regular local rings? There is a complete answer to this, thanks to the

Auslander-Buchsbaum formula: Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. If M has finite projective dimension, then

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R\,.$$

In particular, the only MCM modules of finite projective dimension are free. In further particular, the only MCM modules over a regular local ring are the free ones. This last fact even characterizes regular local rings.

So this is the first hint of why we care about MCM modules over CM rings: they carry homological information about the ring. I'll give two more reasons.

The homological conjectures. In commutative algebra there is a set of conjectures due to Auslander, Bass, Peskine-Szpiro, Serre, and especially Hochster in the 60s through 90s. For example,

Auslander's Nonzerodivisor Conjecture: Suppose M has finite projective dimension and $z \in \mathfrak{m}$ is a nonzerodivisor on M. Then z is a nonzerodivisor on R.

Bass's Question: If a local ring R admits a finitely generated module of finite injective dimension, then R is CM.

New Intersection Theorem: Let $0 \longrightarrow F_k \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$ be a complex of finitely generated free R-modules, with homology of finite length. If $k < d$, then the complex is exact.

And a bunch of others (Direct Summand Conjecture, Canonical Element Conjecture, Serre's Conjectures on Intersection Multiplicities). Many of them are known to have positive answers, due to Peskine-Szpiro and Roberts. But the point is that existence of a MCM module (even an infinitely generated one! so-called "Big CM modules") implies all these conjectures. See Hochster's many expository writeups for more on this.

Here's another reason: MCM modules are an appropriate class of modules to consider as a representation theory. It follows from a result of Bass that we should not consider all modules in general: If a local ring R has finite representation type with respect to all f.g. modules, then R is Artinian.

We therefore say that R has finite CM representation type or finite CM type or just frt if it has only finitely many indecomposable MCM modules up to isomorphism.

Let's again consider low dimensions.

$\underline{\dim R = 0.}$ An Artinian local ring of finite CM type is a principal ideal ring, e.g. $k[t]/(t^n)$ or $\mathbb{Z}/(p^n)$. The commutative situation is thus significantly simpler than the general non-commutative situation.

$\underline{\dim R = 1.}$ One-dimensional rings of finite CM type are characterized by two conditions due to Drozd and Roiter: they have multiplicity at most 3, and the module $\mathfrak{m}\overline{R} + R/R$ is cyclic, where $\overline{R}$ is the integral closure.

$\underline{\dim R = 2.}$ An amazing theorem due to Auslander completely classifies the (complete, equicharacteristic zero) two-dimensional CM local rings of finite CM type: They are precisely those of the form $k[[u, v]]^G$, where G is a finite subgroup of $\mathrm{SL}(2, k)$. Such a ring was earlier proven to have finite CM type by Herzog.

(This theorem is the starting point of a beautiful story called the McKay Correspondence, which I love to tell, but won't fit in today's lecture.)

$\underline{\dim R \geqslant 3}$ There is no classification known. In fact there are only two non-hypersurface examples known (more on hypersurfaces in a moment).

Now two general theorems due to Auslander.

Auslander '87: A CM local ring of finite CM type must be an isolated singularity, that is, every localization away from the maximal ideal is a regular local ring.

Auslander '86: MCM modules over isolated singularities have almost split sequences (AR sequences).

So the representation theory of a CM local ring of finite CM type can be treated in much the same way as the representation theory of a finite-dimensional algebra of finite representation type.

This leads to arguably the biggest success of the "classical phase" of MCM representation theory: AR sequences are used to prove the following classification of hypersurfaces of finite CM type [Buchweitz-Greuel-Schreyer, Knörrer].

Let $R = k[\![x_0, \ldots, x_d]\!]/(f)$ be a hypersurface ring, where k is an algebraically closed field of characteristic zero and $0 \neq f \in (x)^2$. The following are equivalent:

(1) R has finite CM type
(2) R is a simple singularity in the sense of Arnold.
(3) $R \cong k[\![x, y, z_2, \ldots, z_d]\!]/(g(x,y) + z_2^2 + \cdots + z_d^2)$, where $g(x,y)$ is one of the following polynomials, indexed by the ADE Dynkin diagrams:

$$\begin{array}{lll} (A_n): & x^2 + y^{n+1}\ , & n \geqslant 1 \\ (D_n): & x^2 y + y^{n-1}\ , & n \geqslant 4 \\ (E_6): & x^3 + y^4 & \\ (E_7): & x^3 + xy^3 & \\ (E_8): & x^3 + y^5 & \end{array}$$

This classification is a completely explicit one: the indecomposable MCM modules for each of these ADE hypersurface singularities are known: we have presentation matrices for them.

In fact [Knörrer] the hypersurface $R = S/(f)$ has frt if and only if the double branched cover $R^\# = S[z]/(f + z^2)$ does. (The functions $M \mapsto \operatorname{syz}_1^{R^\#}(M)$ and

$N \mapsto N/zN$ are not bijections, but are at most 2-to-1 and miss at most half the target modules, <u>and</u> doing them twice <u>is</u> a bijection between MCM modules over R and $R^{\#\#}$.)

This, together with a lemma (if $S/(f)$ has dimension > 1 and has frt, then f is of the form $g + x_d^2$ with g not involving x_d), reduces the classification to dimension one, where one can use the Drozd-Roiter conditions to do the classification by hand.

Then one wants to produce all the indecomposable MCM modules for the one-dimensional ADE hypersurfaces. This is where AR theory comes in. One needs, for example,

2^{nd} **Brauer-Thrall Theorem:** Let R be an excellent equicharacteristic CM local ring with perfect residue field. Then R has frt if and only if R is an isolated singularity and there is a bound on the ranks (multiplicities) of the indecomposable MCM modules.

(Alternatively, produce all the indecomposable MCM modules for the two-dimensional ones with the McKay Correspondence!)

Finally, a few words about how to describe MCM modules. The answer is particularly satisfactory over hypersurfaces, so assume $R = S/(f)$, where $S = k[\![x_0, \dots, x_d]\!]$ is a $(d+1)$-dimensional power series ring. Let M be a MCM R-module. Then $\operatorname{depth}_R M = d$. We can compute depth over R or S, so $\operatorname{depth}_S M = d$ as well. By the Auslander-Buchsbaum formula, this means $\operatorname{pd}_S M = \operatorname{depth} S - \operatorname{depth} M = 1$.

So M has a free resolution $0 \longrightarrow S^n \xrightarrow{A} S^n \longrightarrow M \longrightarrow 0$, where A is a square matrix over S. Since multiplication by f kills M, it is null-homotopic on the resolution. This means that there is a matrix B of the same size as A so that $AB = f \cdot I_n$. Then $BA = f \cdot I_n$ too, that is, A and B form a <u>matrix factorization of f</u>.

Eisenbud 1980: The functor $(A, B) \mapsto \operatorname{cok} A$ defines an equivalence of categories between (reduced) matrix factorizations of f (up to equivalence) and (stable) MCM modules (up to isomorphism). In particular, the free resolution of M over R is of the form $\cdots \xrightarrow{A} R^n \xrightarrow{B} R^n \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$.

This allows the use of matrix operations to classify MCM modules over hypersurfaces. Among other things!

Part 2. MCM endomorphism rings and n-cluster tilting

For various applications, it would be desirable that the endomorphism ring of a MCM module be again MCM. As an example, a non-commutative crepant resolution of a Gorenstein normal domain is an endomorphism ring $\mathrm{End}_R(M)$ of a reflexive module which has finite global dimension and is MCM.

Unfortunately, this isn't true in general. The most we can say in general is

Lemma. Let R be a CM local ring and M, N finitely generated modules modules with $\operatorname{depth} N \geqslant 2$. Then $\operatorname{depth} \mathrm{Hom}_R(M, N) \geqslant 2$.

Other than this, depth of endomorphism rings is unpredictable. It can go up or down.

Example. Let $(R, \mathfrak{m})$ be a CM loca normal domain of dimension $d \geqslant 2$. Then $\operatorname{depth} \mathfrak{m} = 1$, but $\mathrm{End}_R(\mathfrak{m}) = R$ is MCM.

Example. Let $R = k[[x^2, xy, y^2, yz, z^2, xz]]$, $I = (x^2, xy, xz)R$, and $M = \mathrm{syz}_1^R(I)$. Then M is MCM (depth 3) but $\operatorname{depth} \mathrm{End}_R(M) = 2$.

(Observe that this is one of the two known non-hypersurface CM local rings of finite CM type in dimension $\geqslant 3$. On the other hand, maybe you think that such an example might not exist over a Gorenstein ring. This was a question of Vasconcelos '68.)

Example (Jinnah '74, based on suggestion of Hochster)**.** Put $A = k[x_1, x_2, x_3]/(x_1^3 + x_2^3 + x_3^3)$ and $B = k[a, b]$. Let $R = A \# B$ be the Segre product, defined as a graded ring by $R_n = A_n \otimes_k B_n$. Then R is 3-dimensional, not CM, but has an ideal I of depth 3. Write R as a quotient of a 3-dimensional complete intersection S. Then $\mathrm{Hom}_S(I, I)$ has depth 2.

At least it wasn't easy.

Recall, however, that for representation-theoretic purposes we were most interested in isolated singularities. In that case, there is a natural condition that ensures a Hom-module has good depth.

Lemma. Let R be a CM isolated singularity of dimension d and M, N two MCM R-modules. If $\mathrm{Ext}^i_R(M,N) = 0$ for $i = 1, \dots, r$, then $\operatorname{depth} \mathrm{Hom}_R(M,N) \geqslant r+2$. In particular, if $\mathrm{Ext}^i_R(M,M) = 0$ for $i = 1, \dots, d-2$, then $\mathrm{End}_R(M)$ is MCM. The converse holds as well.

Proof. Resolve $\cdots \mathcal{F}_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$ and apply $\mathrm{Hom}_R(-,N)$. The result,

$$0 \longrightarrow (M,N) \longrightarrow (F_0,N) \longrightarrow \cdots \longrightarrow (F_r,N) \longrightarrow \mathrm{cok} \longrightarrow 0$$

is a complex. If the Exts vanish, it is exact, and each term of the form $\mathrm{Hom}_R(F_i,N)$ is MCM, so $\mathrm{Hom}_R(M,N)$ has depth at least $r+2$ by the Depth Lemma (behavior of depth along exact sequence). The converse uses the Acyclicity Lemma. □

This suggests the following definition. Recall that $\operatorname{add} M = \operatorname{add}_R M$ consists of all modules isomorphic to a direct summand of a direct sum of copies of M.

Definition. Let M be a MCM module and let $n \geqslant 1$. We say that M is <u>n-cluster tilting</u> if

$$\begin{aligned} \operatorname{add} M &= \left\{X \text{ MCM} \mid \mathrm{Ext}^i_R(M,X) = 0 \text{ for } i = 1, \dots, d-1\right\} \\ &= \left\{X \text{ MCM} \mid \mathrm{Ext}^i_R(X,M) = 0 \text{ for } i = 1, \dots, d-1\right\} \end{aligned}$$

This means that if R is an isolated singularity and M is $(d-1)$-cluster tilting, then $\mathrm{Ext}^{1,\dots,d-2}(M,M) = 0$, so that $\mathrm{End}_R(M)$ is MCM.

<u>$n=1$</u> is no condition on the modules X, so M is 1-cluster tilting iff $\operatorname{add} M$ contains every MCM module. In other words, there is a 1-cluster tilting module iff R has finite CM representation type. In this situation, the endomorphism ring $\mathrm{End}_R(M)$ of any MCM generator is called the <u>Auslander algebra</u> of R.

We therefore call the endomorphism ring of an n-cluster tilting module an <u>n-Auslander algebra</u> of R.

Subcategories of the form $\operatorname{add} M$, where M is an n-cluster tilting module, are called n-cluster tilting subcategories.

There is a "higher-dimensional AR theory" for n-cluster tilting subcategories [Iyama, Herschend, Oppermann, in various combinations; here I am following a lecture of Iyama from Guanajuato in 2012]:

Theorem. *Let R be an isolated singularity. Let M be an n-cluster tilting module and let $\mathcal{C} = \operatorname{add} M$ be the corresponding n-cluster tilting subcategory.*

(1) There is an equivalence $\tau_n \colon \underline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}$ ("higher AR translate").
(2) There is a functorial isomorphism

$$\underline{\operatorname{Hom}}_R(X, Y) \cong \operatorname{Ext}^n_R(Y, \tau_n X)^\dagger$$

for all $X, Y \in \mathcal{C}$ ("higher AR duality"). Here $-^\dagger$ is Matlis duality.
(3) For $X \in \mathcal{C}$ indecomposable and not projective, there is an exact sequence

$$0 \longrightarrow \tau_n X \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow X \longrightarrow 0$$

with each Y_i in $\mathcal{C}$, and the map to X (resp. from $\tau_n X$) is a right (left) almost split morphism in $\mathcal{C}$ ("n-AR sequence").

Some comments on the pieces of the Theorem. The higher AR translate τ_n is defined by $\tau_n(-) = (\Omega^d \operatorname{Tr} \Omega^{n-1}(-))^\vee$, where $-^\vee = \operatorname{Hom}_R(-, \omega)$ is the duality into the canonical module. In particular, in the special case $n = d-1$, τ_{d-1} coincides with the Nakayama functor $\nu(-) = (-)^{\vee *}$.

As in the "classical" case of AR theory, the n-almost split sequence is the (unique up to isomorphism) socle element of $\operatorname{Ext}^n(X, \tau_n X) \cong \operatorname{End}_R(X)^\dagger$.

So do there exist any examples of n-cluster tilting modules?

Example. Let $S = k[[x_1, \ldots, x_d]]$ with $d \geqslant 2$, and let $G \subset \operatorname{SL}_d(k)$ be a finite subgroup with order invertible in k. Set $R = S^G$, the invariant subring. Then

R is a Gorenstein normal domain of dimension d. Let's assume further that R is an isolated singularity, that is, no non-identity $\sigma \in G$ has 1 for an eigenvalue. Then [Auslander] $\mathrm{End}_R(S) \cong S\#G$ is the twisted group ring, and in particular is MCM as S-module. It follows that S is a $(d-1)$-cluster tilting R-module.

If $d = 2$ this means S is 1-cluster tilting, so that $\mathrm{add}_R\, S$ contains all MCM R-modules.

Example (BIKR)**.** Let $R = k[[x,y]]/(f(x,y))$, so that R is the coordinate ring of a plane curve singularity. Assume R has an isolated singularity, i.e. is reduced, i.e. f has no repeated factors. Factor $f = f_1 f_2 \cdots f_n$ into irreducibles. Then R has a 1-cluster tilting module if and only if each $f_i \notin (x,y)^2$. In this case there are $n!$ 1-cluster tilting modules. Here is one:

$$M_{\mathrm{id}} = S/(f_1) \oplus S/(f_1 f_2) \oplus \cdots \oplus S/(f_1 f_2 \cdots f_n)$$

And for every permutation in S_n you get another one in the obvious way.

Example. In usual AR theory, for 2-dimensional rings, the fundamental sequence plays a central role. Let R be a 2-dimensional complete local normal domain with canonical module ω. Then ω is characterized by $\mathrm{Ext}^i_R(R/\mathfrak{m}, \omega) = \delta_{i,2} k$. In particular there is a unique extension

$$\varphi \colon 0 \longrightarrow \omega \longrightarrow E \longrightarrow \mathfrak{m} \longrightarrow 0\,.$$

Auslander: Every AR sequence over R is obtained from the fundamental sequence: the AR sequence for M is $\mathrm{Hom}(M^*, \varphi)$.

Iyama: Let R be d-dimensional. If R has a $(d-1)$-cluster tilting module, then it has an n-fundamental sequence playing the same role.

Back to the n-Auslander algebras for a moment. Recall the classical result of Auslander: if Λ is a finite-dimensional algebra of finite representation type, with representation generator M, then the "Auslander algebra" $\mathrm{End}_\Lambda(M)$ has global dimension $\leqslant 2$ and dominant dimension $\geqslant 2$. Moreover the correspondence $\Lambda \mapsto \Gamma$ defines a bijection between Morita-equivalence classes of finite-dimensional algebras and algebras with $\mathrm{gldim} \leqslant 2$ and $\mathrm{domdim} \geqslant 2$.

Iyama: There is a bijection between CM local rings R of dimension d with an n-cluster tilting module and R-orders Γ with isolated singularity, $\mathrm{gldim}\,\Gamma \leqslant n+1$, and such that Γ and Γ° both satisfy a version of $\mathrm{domdim} \geqslant n+1$ (projective dimension of I^i in a minimal injective resolution is at most $n+1-d$ for $i < n+1-d$).

In particular the case $n = 1$ answers a question of M. Artin, to characterize homologically the endomorphism rings $\Gamma = \mathrm{End}_{\Lambda}(M)$ of representation generators M for orders Λ of finite CM representation type. Specifically, they are the isolated singularities of global dimension d and depth at least 2, such that there exists an idempotent $e \in \Gamma$ such that $e\Gamma$ is MCM, $\overline{\Gamma} = \Gamma/\Gamma e \Gamma$ is Artinian, and $\mathrm{pd}_{\Gamma} X \leqslant 2$ holds for any $\overline{\Gamma}$-module X.