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"MCM modules and Matrix Factorizations"

This is a preparatory talk for my later research talk (and Buchweitz's as well).

<u>Observe</u>: The polynomial $x^2+y^2+z^2$ is irreducible over any field of characteristic $\neq 2$. In particular it can't be factored nontrivially.

However,

$$\begin{bmatrix} x^2+y^2+z^2 & 0 \\ 0 & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} x & -y+iz \\ y+iz & x \end{bmatrix} \begin{bmatrix} x & y+iz \\ -y+iz & x \end{bmatrix}$$

We call this a <u>matrix factorization</u> of ~~the~~ $x^2+y^2+z^2$.

<u>Def</u> Let S be a (regular (local)) ring and $f \in S$. A <u>matrix factorization</u> of f is a pair

$$\varphi : G \longrightarrow F \quad , \quad \psi : F \longrightarrow G$$

of homom's between f.g. free S-modules, satisfying

$$\psi\varphi = f \cdot \mathrm{id}_G \quad , \quad \varphi\psi = f \cdot \mathrm{id}_F$$

Choosing bases to represent φ, ψ as matrices, we get $\psi\varphi = f I_n$, $\varphi\psi = f I_n$.

non-zerodivisor

MFs and modules: Assume $f \in S$ is nzd.
Then φ and ψ must be 1-1,
so we get exact seq's of S-modules

$$0 \to G \xrightarrow{\varphi} F \to \operatorname{cok}\varphi \to 0$$
$$0 \to F \xrightarrow{\psi} G \to \operatorname{cok}\psi \to 0.$$

Notice The element f knocks F into $f(F) = \varphi\psi(F)$, so annihilates $\operatorname{cok}\varphi$. Therefore $\operatorname{cok}\varphi$ is naturally a module over the hypersurface ring $R = S/(f)$

Recall: ① A (f.g.) module over a local ring A is called Maximal Cohen-Macaulay (MCM) if

$$\operatorname{depth}_A M = \dim A$$

(the max possible value).

② The Auslander-Buchsbaum formula says if $\operatorname{pd}_A M < \infty$ then

$$\operatorname{pd}_A M + \operatorname{depth}_A M = \operatorname{depth} A.$$

So: As an S-module, $\operatorname{cok}\varphi$ has proj. dim 1, hence depth equal to $\operatorname{depth} S - 1 = \dim S - 1$. That is the same as $\dim R$

$\Rightarrow M$ is a MCM R-module.

Conversely: Any MCM R-module M has proj. dim 1 over S, so a resolution of the form

$$0 \to G \xrightarrow{X} F \to M \to 0.$$

Multiplication by f kills M, so is null-homotopic on this, so $\exists\ Y : F \to G$ s.t.

$$XY = f \cdot \mathrm{id}_F$$

Since X is injective, $YX = f \cdot \mathrm{id}_G$ too so X, Y is a MF

Theorem [Eisenbud '80] Let S be a regular local ring, $f \in S$ nzd, $R = S/(f)$. Then

$$(\varphi, \psi) \longleftrightarrow \operatorname{cok} \varphi$$

defines a bijection between reduced MFs of f (up to a natural equiv rel) and MCM R-modules without free $\oplus$-ands.

(reduced means no $\oplus$-ands $\cong (f, 1)$ or $(1, f)$.)

Let M be a MCM module over $R = S/(f)$. We know $M = \operatorname{cok} \varphi$ for a MF (φ, ψ).

Since $\varphi\psi = f \cdot \mathrm{id} = \psi\varphi$, over R

we have $\operatorname{im}\overline{\varphi} = \ker\overline{\psi}$, $\operatorname{im}\overline{\psi} = \ker\overline{\varphi}$.
So
$$\cdots \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \longrightarrow M \longrightarrow 0$$
is the free resolution of M over R.

<u>Consequence</u>: The minimal free res'n of a MCM module over a hypersurface ring is 2-periodic. <u>All</u> min'l free res'ns over R are <u>eventually</u> 2-periodic.

So, for example, $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i+2}_R(M,N)$ if M is MCM, R hypersurface.

<u>Conversely</u> [Tate] Gulliksen If every minimal free resn over a local ring has bounded Betti #s (ranks of free mods) then R is a hypersurface ring

☆ So a local ring is a hypersurface ring <u>iff</u> every min'l free res'n is eventually 2-periodic.

(BTW, if f is irreducible and (φ, ψ) is a MF of size n, then $\det\varphi = f^k$, $\det\psi = f^{n-k}$ and one can show $k = \operatorname{rank} M$.)

Application: Classification of Hypersurface Rings of Finite Representation Type.

Say a CM local ring R has f.r.t. if it has only finitely many indecomposable MCM modules up to $\cong$.

For a hypersurface this means only fin. many indec MFs up to equiv.

Thm [Buchweitz-Greuel-Schreyer] If a hypersurface $R = S/(f)$ has f.r.t., then the set
$$c(f) \overset{\text{def}}{=} \{\text{ideals } I \subseteq S \mid f \in I^2\}$$
is finite.
(Say f is simple.)

Pf sketch If M is any MCM R-mod w/ MF (φ, ψ), set
$$J(M) = I_1(\varphi) + I_1(\psi).$$
Then
$$f \in I_1(\varphi\psi) \subseteq I_1(\varphi) I_1(\psi) \subseteq J(M)^2$$
so $J(M) \in c(f)$.

OTOH, for any ~~I(f)~~ ideal $J \in c(f)$, we can define a MF: if
$$f = \sum_{i=1}^{r} x_i y_i \qquad x_i, y_i \in J,$$

with $J = I_1(\varphi) + I_1(\psi)$.

$r = 1$: take $\varphi = (x_1)$, $\psi = (y_1)$

$r \geq 2$, inductively: take

$$(\varphi, \psi) = \left(\begin{bmatrix} x_r I & \varphi_{r-1} \\ \psi_{r-1} & -yI \end{bmatrix}, \begin{bmatrix} y_r & \varphi_{r-1} \\ \psi_{r-1} & -x_r \end{bmatrix} \right)$$

Since R has FRT, any (φ, ψ) is isom to a $\oplus$ of copies of some finite list of indec MFs

$$(\sigma_1, \tau_1), \ldots, (\sigma_n, \tau_n)$$

Then $J(\varphi, \psi) = \sum J(\sigma_i, \tau_i)$ is a sum of finite subsets of that finite list of ideals. So there are only fin. many J's hence $c(f)$ is finite. ✓ $\square$.

<u>FACT</u> The simple $f \in k[[x, y]]$ (so $\dim R = 1$) are not hard to classify. They are

$\boxed{k = \bar{k} \\ \text{char } k = 0}$

A_n	$x^2 + y^{n+1}$
D_n	$x^2 + xy^{n-1}$
E_6	$x^3 + y^4$
E_7	$x^3 + xy^3$
E_8	$x^3 + y^8$.

Thm These are the 1-dim'l hyp's of FRT.

So now we need an inductive tool to relate FRT in dim n to dim $n+1$.

Next time: Knörrer's construction.