A GENERALIZATION OF KNÖRRER PERIODICITY WITH AN APPLICATION TO NONCOMMUTATIVE HYPERSURFACES

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Let S be a regular local ring of dimension $d+1$, and let f be a non-zero non-unit of S. Let $R = S/(f)$ be the d-dimensional hypersurface ring. Fix an integer $n \geqslant 2$ and set $R^{\#} = S[y]/(f+y^n)$. We refer to $R^{\#}$ as the *(n-fold) branched cover* of R.

We consider maximal Cohen-Macaulay (MCM) modules over R and $R^{\#}$. Our starting point is the following theorem of Knörrer and its generalization by Herzog-Popescu. Write Ω for the syzygy operator.

Theorem ([Knö87]). *Suppose that $S = k[\![x_0, \dots, x_d]\!]$ and the characteristic of k is not equal to 2. Assume $n = 2$, so that $R^{\#} = S[y]/(f + y^2)$. Then for any MCM $R^{\#}$-module N, we have*

$$\Omega_{R^{\#}}(N/yN) \cong N \oplus \Omega_{R^{\#}}(N)\,.$$

Theorem ([HP97, Theorem 2.6]). *Suppose that $S = k[\![x_0, \dots, x_d]\!]$ and the characteristic of k does not divide n. Let $n \geqslant 2$ be arbitrary. Then for any MCM $R^{\#}$-module N, we have*

$$\Omega_{R^{\#}}(N/y^{n-1}N) \cong N \oplus \Omega_{R^{\#}}(N)\,.$$

We consider the nested sequence of full subcategories of $R^{\#}$-modules of the form $\Omega_{R^{\#}}(M)$, where M is annihilated by y^k, for $k = 1, \dots, n$. Specifically, set $A_k = S[y]/(f, y^k)$ for $1 \leqslant k \leqslant n$, and define

$$\Sigma_k = \operatorname{add}\left\{\Omega_{R^{\#}}(M) \,|\, M \text{ is a MCM } A_k\text{-module}\right\}\,.$$

We have $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \mathrm{MCM}(R^{\#})$. Observe that the results of Knörrer and Herzog-Popescu above imply $\Sigma_1 = \mathrm{MCM}(R^{\#})$ when $n = 2$, and $\Sigma_{n-1} = \mathrm{MCM}(R^{\#})$ for arbitrary n, respectively.

Our main result is the following.

Theorem. *The subcategories $\Sigma_1, \dots, \Sigma_n$ are functorially finite in $\mathrm{MCM}(R^{\#})$, that is, every MCM $R^{\#}$-module N admits both left and right approximations by modules in Σ_k for each k. Namely, there are short exact sequences*

$$0 \longrightarrow \Omega_{R^{\#}}(N) \longrightarrow \Omega_{R^{\#}}(N/y^kN) \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow \Omega_{R^{\#}}(\Omega_{R^{\#}}(N)/y^k\Omega_{R^{\#}}(N)) \longrightarrow \Omega_{R^{\#}}(N) \longrightarrow 0$$

which are right and left Σ_k-approximations of N, respectively.

As an application of the Theorem, we extend a result of Iyama, Leuschke, and Quarles to the effect that the endomorphism ring of a representation generator for the MCM modules over a simple (ADE) hypersurface singularity has finite global dimension. When $f \in k[\![x_0, \dots, x_d]\!]$ defines a simple singularity of dimension $d \geqslant 1$, the polynomial $f + y^n$ does not in general define a simple singularity for $n \geqslant 3$, so that in particular the branched cover does not have a representation generator. However we show that the n-fold branched cover admits a MCM module M such that $\Lambda = \mathrm{End}_{R^{\#}}(M)$ behaves like a "non-commutative hypersurface ring," in the sense that Λ is Iwanaga-Gorenstein and every finitely generated Λ-module has a projective resolution which is eventually periodic of period at most 2.

References

[HP97] Jürgen Herzog and Dorin Popescu, *Thom-Sebastiani problems for maximal Cohen-Macaulay modules*, Math. Ann. **309** (1997), no. 4, 677–700. MR 1483829

[Iya07] Osamu Iyama, *Auslander correspondence*, Adv. Math. **210** (2007), no. 1, 51–82. MR 2298820

[Knö87] Horst Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164. MR 877010

[Leu07] Graham J. Leuschke, *Endomorphism rings of finite global dimension*, Canad. J. Math. **59** (2007), no. 2, 332–342. MR 2310620

[OP00] Liam O'Carroll and Dorin Popescu, *On a theorem of Knörrer concerning Cohen-Macaulay modules*, J. Pure Appl. Algebra **152** (2000), no. 1-3, 293–302, Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998). MR 1784003 (2001g:13020)

[Qua05] Christopher L. Quarles, *Krull-Schmidt Rings & Noncommutative Resolutions of Singularities*, Master's thesis, University of Washington, 2005.

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