

Branched Covers and Matrix Factorizations

Graham Leuschke, Tim Tribone

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Matrix factorizations

Observe that $f = xy + z^2$ is **irreducible** over any field k . However,

$$\begin{bmatrix} xy + z^2 & \\ & xy + z^2 \end{bmatrix} = \begin{bmatrix} x & z \\ -z & y \end{bmatrix} \begin{bmatrix} y & -z \\ z & x \end{bmatrix}.$$

We call such a thing a matrix factorization of f .

Definition (Eisenbud '80)

Let S be a (regular local) ring and $f \in S$. A **matrix factorization (MF)** of f is a pair of $n \times n$ matrices (φ, ψ) such that

$$\varphi\psi = f \cdot I_n = \psi\varphi.$$

MFs and MCM modules

Suppose that S is a regular local ring, $f \neq 0$ and (φ, ψ) is a matrix factorization of f . Then φ and ψ must be injective, so we get a short exact sequence of S -modules

$$0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow \operatorname{coker} \varphi \longrightarrow 0$$

and similarly for ψ .

Now use the Auslander-Buchsbaum formula to see that

Fact

*The cokernels of φ and ψ are **maximal Cohen-Macaulay (MCM)** modules over the hypersurface ring $R = S/(f)$. This means $\operatorname{depth} \operatorname{coker} \varphi = \dim R$.*

MFs and MCM modules, cont'd

Example

- $(f, 1) \mapsto \operatorname{coker} f = R$
- $(1, f) \mapsto \operatorname{coker} 1 = 0$

Call a matrix factorization **reduced** if it has no unit entries.

Theorem (Eisenbud, '80)

The correspondence

$$(\varphi, \psi) \longleftrightarrow \operatorname{coker} \varphi$$

defines a bijection between reduced MFs of f (up to a natural equivalence relation) and MCM R -modules with no free direct summands (up to isomorphism).

Example

The only indecomposable MCM modules over $R = k[[x]]/(x^2)$ are the ring itself and the residue field; equivalently, the only indecomposable MFs of $x^2 \in k[[x]]$ are $(x^2, 1)$, (x, x) , and $(1, x^2)$.

Example (Buchweitz-Greuel-Schreyer, '87)

The only indecomposable MFs of $x^2 \in k[[x, y]]$ are

$$(x^2, 1), \quad (x, x), \quad (1, x^2)$$

$$\left(\begin{bmatrix} x & y^n \\ & x \end{bmatrix}, \begin{bmatrix} x & -y^n \\ & x \end{bmatrix} \right).$$

This gives a complete classification of MCM modules over $R = k[[x, y]]/(x^2)$.

More interesting application

Definition (Yoshino)

Given two MFs (φ, ψ) of f and (φ', ψ') of g in separate sets of variables, define

$$(\varphi, \psi) \hat{\otimes} (\varphi', \psi') = \left(\begin{bmatrix} \psi & -\varphi' \\ \psi' & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & \varphi' \\ -\psi' & \psi \end{bmatrix} \right)$$

the **tensor product** of MFs. It is a MF of $f + g$.

This has nothing to do with $\operatorname{coker} \varphi \otimes \operatorname{coker} \varphi'$.

This gives a non-obvious mapping

$$\operatorname{MCM}(S/(f)) \times \operatorname{MCM}(S'/(g)) \longrightarrow \operatorname{MCM}(S''/(f + g)).$$

Special case, the double branched cover

When $g = z^2 \in k[[z]]$, we get

$$R = S/(f) \quad \text{and} \quad R^\# = S[[z]]/(f + z^2),$$

defined by

$$\begin{aligned} (\varphi, \psi) &\longmapsto (\varphi, \psi) \hat{\otimes} (z, z) \\ &= \left(\begin{bmatrix} \psi & -z \\ z & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & z \\ -z & \psi \end{bmatrix} \right) \end{aligned}$$

One can show that this corresponds to

$$\text{coker } \varphi \longmapsto \text{syz}^{R^\#}(\text{coker } \varphi).$$

We also have a reverse functor, killing z .

Knörrer's results

$$R = S/(f), R^\# = S[[z]]/(f + z^2)$$

Theorem (Eisenbud '80, Knörrer '87)

Assume $\text{char } k \neq 2$. Then the following are equivalent:

1. There are only finitely many indecomposable MCM R -modules up to $\cong$ (*finite CM type*).
2. There are only finitely many indecomposable MFs of f (*finite MF type*).
3. There are only finitely many indecomposable MCM $R^\#$ -modules up to $\cong$.
4. There are only finitely many indecomposable MFs of $f + z^2$.

Classification of finite CM type

Theorem (Buchweitz-Greuel-Schreyer, Knörrer '87)

Let k be an algebraically closed field of characteristic zero and let $R = k[[x_1, \dots, x_r]]/(f)$, where $f \in (x_1, \dots, x_r)^2$ is nonzero. Then R has finite CM type if and only if R is a **simple hypersurface singularity**, that is, after a change of variables f is among the following:

$$(A_n) \quad x^2 + y^{n+1} + x_3^2 + \cdots + x_r^2, \text{ for } n \geq 1$$

$$(D_n) \quad x^2 y + y^{n-1} + x_3^2 + \cdots + x_r^2, \text{ for } n \geq 4$$

$$(E_6) \quad x^3 + y^4 + x_3^2 + \cdots + x_r^2$$

$$(E_7) \quad x^3 + xy^3 + x_3^2 + \cdots + x_r^2$$

$$(E_8) \quad x^3 + y^5 + x_3^2 + \cdots + x_r^2$$

Key Ingredient

Knörrer related MFs of f to the **skew group algebra** $R^\#[\sigma]$, where σ is the automorphism of $R^\# = S[[z]]/(f + z^2)$ that fixes S and sends $z \mapsto -z$.

Elements of $R^\#[\sigma]$ are

$$a \cdot 1 + b \cdot \sigma,$$

with $a, b \in R^\#$ and

$$\sigma a = \sigma(a)\sigma.$$

Theorem (Knörrer '87)

*There is an **equivalence***

$$\mathrm{MF}(f) \simeq \mathrm{MCM}(R^\#[\sigma])$$

(MCM $R^\#$ -modules with an action of σ).

From now on, fix an integer $d \geq 2$ and assume $\text{char } k \nmid d$.

Definition

A d -fold matrix factorization of f is a d -tuple of square matrices $(\varphi_1, \dots, \varphi_d)$ with

$$\varphi_1 \cdots \varphi_d = f \cdot I_n.$$

Get a (exact, Frobenius) category $\text{MF}^d(f)$.

First (?) considered by Backelin-Herzog-Sanders '06 for homogeneous f of degree d .

First observations

- Given any “normal” MF (φ, ψ) , we can inflate it to $(\varphi, \psi, 1, 1, \dots, 1)$ for any d . These are boring.
- There exist non-boring ones: $(x, y, z) \in \text{MF}^d(xyz)$.
- We can also collapse: $(\varphi_1, \dots, \varphi_d) \mapsto (\varphi_1, \varphi_2 \cdots \varphi_d)$.
Collapsing corresponds to an **extension** of MCM modules.

Example

If $(\varphi_1, \varphi_2, \varphi_3) \in \text{MF}^3(f)$, then there is a short exact sequence

$$0 \longrightarrow \text{coker } \varphi_2 \longrightarrow \text{coker}(\varphi_2 \varphi_3) \longrightarrow \text{coker } \varphi_3 \longrightarrow 0$$

of R -modules.

The d -fold Branched Cover

Question

The modules corresponding to d -fold MFs form a subcategory of $\text{MCM}(R)$. Which ones are they?

Answer

We relate them to the d -fold branched cover defined by $f + z^d$.

Theorem (Tribone '21)

There is an equivalence

$$\mathrm{MF}^d(f) \simeq \mathrm{MCM}(R^\#[\sigma]),$$

where now σ is the order- d automorphism of $R^\# = S[[z]]/(f + z^d)$ fixing S and sending $z \mapsto \omega_d z$.

Corollary

The category $\mathrm{MF}^d(f)$ has the Krull-Remak-Schmidt uniqueness property for direct-sum decompositions, and has enough projectives.

Tribone gives formulas for the syzygy of a d -MF, similar to previous ones by Iyama-Kato-Miyachi for “ d -complexes”.

Example

Let $f = xyz \in k[[x, y, z]]$ and consider

$X = (x, y, z) \in \text{MF}^3(xyz)$. Then the syzygy of X is

$$\left(\begin{bmatrix} -y & -yz \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -z & -xz \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -x & -xy \\ 1 & 0 \end{bmatrix} \right)$$

Note that this is an indecomposable 3-MF, even though it is not reduced! More on this later.

Corollary (Tribone)

The formulas for syzygies indicate that $R^\#[\sigma]$ “behaves like a hypersurface”, in that all minimal projective resolutions are eventually periodic of period at most 2.

Extending Knörrer, cont'd

Theorem (L-Tribone '22)

The following are equivalent:

- (1) *There are only finitely many indecomposable d -fold MFs of f .*
- (2) *The d -fold branched cover $R^\# = S[[z]]/(f + z^d)$ has finite CM type.*

Proof Idea.

Given $N \in \text{MCM}(R^\#)$, let $\varphi: N \rightarrow N$ be the S -linear map given by multiplication by z . Since $z^d = -f$, get a d -fold MF $N^\flat = (\varphi, \dots, \varphi) \in \text{MF}^d(f)$ (up to sign).

Given a d -MF $(\varphi_1, \dots, \varphi_d)$ of size n , define a MCM $R^\#$ -module structure on R^{nd} with z -action given by the φ_i .

These aren't inverses, but the composition is understandable. $\square$

Finite d -MF Type

Theorem (L-Tribone)

Assume $d > 2$. Let k be an algebraically closed field of characteristic zero and let $f \in S = k[[y, x_2, \dots, x_r]]$. Then f has finitely many indecomposable d -fold MFs if and only if, after a change of variables, f and d are among:

$$(A_1) \quad y^2 + x_2^2 + \dots + x_r^2, \text{ for any } d > 2$$

$$(A_2) \quad y^3 + x_2^2 + \dots + x_r^2, \text{ for } d = 3, 4, 5$$

$$(A_3) \quad y^4 + x_2^2 + \dots + x_r^2, \text{ for } d = 3$$

$$(A_4) \quad y^5 + x_2^2 + \dots + x_r^2, \text{ for } d = 3$$

For each of these, we can write down the indecomposable d -MFs explicitly using, for example, the classification of MCMs over $z^d + y^2 + x_2^2 + \dots + x_r^2$.

Reduced d -MFs

Fact

If (φ, ψ) is a 2-fold matrix factorization and any entry of either matrix is a unit, then either $(1, f)$ or $(f, 1)$ is a direct summand.

*Equivalently, (φ, ψ) is **reduced** if and only if $\text{coker } \varphi$ and $\text{coker } \psi$ have no free direct summands.*

We already saw this is not true for $d > 2$.

Example

$$X = \left(\begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}, \begin{bmatrix} 0 & y \\ x^2 & -x \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \right)$$

is an indecomposable 3-fold matrix factorization of x^2y .

So which MCM modules correspond to the reduced d -MFs?

Lemma

Let N be an MCM $R^\#$ -module and assume that $f + z^d$ is irreducible (so $R^\#$ is a domain). Then

$$\mu_{R^\#}(N) \leq d \cdot \operatorname{rank}_{R^\#}(N),$$

and equality holds if and only if the S -linear map $z: N \rightarrow N$, considered as a matrix over S , is reduced.

Proof Idea.

The S -module N is free, and the map $z: N \rightarrow N$ is a free presentation of the S -module N/zN . It's a minimal presentation if and only if the matrix is reduced. $\square$

Fact/Definition (Brennan-Herzog-Ulrich '87)

An MCM module M over a domain A always satisfies

$$\mu_A(M) \leq e(A) \cdot \operatorname{rank}_A(M),$$

where $e(A)$ is the multiplicity. If equality holds we say M is an **Ulrich module** (or a **maximally generated MCM module**).

Corollary

Assume that $\operatorname{ord}(f) \leq d$ (so that the multiplicity of $R^\#$ is equal to d). Then an MCM $R^\#$ module N is Ulrich if and only if the corresponding d -MF $N^\flat$ is reduced.

Using the classification of hypersurfaces of finite d -MF type, together with a classification of Ulrich modules over $k[[x, z]]/(x^{n+1} + z^n)$ [Herzog-Ulrich-Backelin '91], we can show:

Example

For $n \geq 4$, the polynomial $x^{n+1} \in k[[x]]$ has infinitely many indecomposable matrix factorizations with n factors, but only finitely many that are reduced, namely $(x^2, x, \dots, x)$ and its shifts.

Thank you!